Left Ideals of Γ-Seminearrings

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Abstract: This paper aims is to study the left duo of Γ-seminearrings. We prove some of the salient features about the left duo of Γ-seminearrings We also characterise such a Γ-seminearring.

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1. Introduction

Hoorn and Rootselaar[1] introduced a more general algebraic structure than a nearring or a semiring is knows as seminearring. Consider a non empty set as a seminearring with two binary operations and such that $(R, +)$ and $(R, .)$ represents a semigroup with one distributive law. The role of seminearring structure applied in many places of theoretical computer science, viz. algebra communicating processes, theory of automata and also seen in semigroup mapping and reversible computation models. Further studuies in cryptographic theories and problems seen in [2]. Let R be an additive semigroup. Let M(R) = $\{\alpha : R \to R\}$ be the set of all mappings of R into itself, with the operations of pointwise addition: $r(\alpha + \beta) = r\alpha + r\beta \ \forall \ r \in R$, and multiplication given by composition of maps: $r(\alpha\beta) = (r\alpha)\beta \ \forall \ r \in R$. Then (M(R), +, .) is a seminearring.

In 1964, Nobusawa introduced Γ-rings. Γ-nearrings defined in 1984 by Satyanarayana and Rao studied Γ-semirings in 1995. It is known that Γ-rings and Γ-nearrings are generalisations of each other. Γ-rings, Γ-nearrings and Γ-semirings are the generalisations of rings, nearrings and semirings respectively.

Manikandan and Perumal[3] introduced and studied the properties of left duo seminearrings. A left ideal of a seminearring R need not be a right ideal in general. This authors concentrate on those seminearrings which exhibit this property.

In this article, we extend this concept to study the left duo of Γ-seminearrings. We prove some of the salient features about the left duo of Γ-seminearrings when admitting mate functions. We also characterise such a Γ-seminearring.

2. Preliminaries

We consider some basic definitions related to seminearrings and Γ-seminearrings used in subsequent sections.

Definition:

A semi-group M is a non-empty set equipped with a binary operation ·, which is associative.

Definition:

Let M and Γ be two non-empty sets. Then M is called a Γ-semigroup if it satisfies

(i) $x\alpha y \in M$ (ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in M$, $\alpha, \beta \in \Gamma$.

Definition:

A semigroup M is said to be commutative if $ab = ba$, for all $a, b \in M$.

Definition:

A Γ-semigroup M is said to be commutative if $a\alpha b = b\alpha a$, for all $a, b \in M$ and $\alpha \in \Gamma$.

Defintion:

A non empty set (R, +, .) with two binary operations addition and multiplication is called seminearring if it is satisfying the following,

(i) (R, +) is a semigroup,

(ii) (R, .) is a semigroup,

(iii) $(a + b)c = ac + bc$ for all $a, b, c \in R$.

Definition:

Let (R, +) be a semigroup and Γ a nonempty set. Then R is called a Γ-seminearring if there exists a mapping from $R \times \Gamma \times R \rightarrow R$, written (a, Γ, b) by aΓb such that it satisfies the following axioms for all $a, b, c \in R$ and $\Gamma, \beta \in \Gamma$:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$ for all $a, b, c \in R$ and $\alpha \in \Gamma$, (ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in R$ and $\alpha, \beta \in \Gamma$.

In the above case, we call R= (R, Γ) a Γ-seminearring.

Definition:

A nonempty subset A of a Γ-seminearring R is a sub Γ-seminearring of R if $a + b \in A$ and $a\alpha b \in A$ for all $a, b \in A$ and $\alpha \in \Gamma$.

Definition:

A Γ-seminearring R is said to have zero element if there exists an element $0 \in R$ such that $0 + x = x = x + 0$ and $0\alpha x = 0x\alpha = 0$, for all $x \in R$ and $\alpha \in \Gamma$.

Definition:

A ring R is called (Von Neumann) regular if for all $a \in R$ there exists $r \in R$ such that $a = ara$.

Definition:

Let R be a Γ-seminearring. An element $a \in R$, is said to be regular element of R if there exist $r \in R$, $\alpha, \beta \in \Gamma$ such that $a = a\alpha r\beta a$.

Definition:

Let R be a Γ-seminearring. If every element of R, is a regular element of R, then R is said to be regular Γ-seminearring R.

Note: Regular rings have been extensively studied in the literature. The concept has been naturally extended to nearrings and seminearrings. The $r \in R$ such that $a = ara$ need not be unique. This leads to the concept of mate functions. A mate function is a self-map $\varphi : R \rightarrow R$ such that $a = a\varphi(a)a$ for all $a \in R$[4, 5]. It is clear that R admits mate function if and only if it is regular.

Definition:

A seminearring R is a left(right) normal if for every element a in R, we have $a \in Ra$ ($a \in aR$) and normal means both normal (left and right).

Definition:

A seminearring R is a left(right) normal if for every element a in R, we have a ∈ RΓa (a ∈ aΓR) and normal means both normal (left and right).

Definition:

There exists a ∈ R then a idempotent if $a^2 = a$ (set of all idempotents of R is E)

Definition:

Let R be a Γ-seminearring. An element a ∈ R is said to be idempotent of R if there exists α ∈ Γ such that a = aαa.

Definition:

If left (right) ideal A is closed with respect to addition and RA ⊆ A (AR ⊆ A), where A is non empty subset. If A is a left as well as a right ideal then it is two sided ideal (or an ideal).

Definition:

A subsemigroup I of R is called an ideal of (R, Γ), if IΓR ⊆ I and RΓI ⊆ I, where by IΓR we mean the set {xΓr | x ∈ I, r ∈ R, Γ ∈ Γ}. If R is a Γ-seminearring with zero element, then it is easy to verify that every ideal I of (R, Γ) has the zero element.

Definition:

A nonempty subset I of S is called a left (right) ideal of S if (I, +) is a subsemigroup of (S, +) and rαx ∈ I (xαr ∈ I) for all r ∈ S, x ∈ I and α ∈ Γ. If I is both a left and a right ideal of S, then I is called an ideal of S. It is obvious that every left (right) ideal of S is a sub Γ-seminearring of S.

Definition:

An ideal I of Γ-seminearring R is a prime if P and Q of R, PΓQ ⊂ I implies either P ⊂ I or Q ⊂ I.

Definition:

An ideal I of Γ-seminearring R is a completely prime if for t, r ∈ R, tΓr ∈ I implies t ∈ I or r ∈ I.

Definition:

I is known as a semiprime ideal of Γ-seminearring R, if aΓa ⊂ I ⇒ a ⊂ I.

Example:

Let R = (Z^+, +) be the semigroup of non-negative integers and let Γ = ($2Z^+$, +) be the semigroup of even positive integers. Then, R is a Γ-seminearring.

Example:

Let R be the additive semigroup of all m×n matrices over the set of all non-negative integers and Γ be the additive semigroup of all n × m matrices over the same set. Then, we can verify that R is a Γ-seminearring, where aΓb is the usual matrix product, for any a, b ∈ R and Γ ∈ Γ.

Example:

Let R_1 and R_2 be two additive semigroups. Let R be the additive semigroup of all homomorphisms from R_1 to R_2 and Γ be the additive commutative semigroup of all homomorphisms from R_1 to R_2. Then R is a Γ-seminearring, where aΓb denotes the usual composition of homomorphisms, a, b ∈ R and Γ ∈ Γ.

3. Main Results

We start this part with left duo structure possess the special structure of Γ-seminearring.

Proposition 3.1.

Every left duo of Γ-seminearring with mate functions is left bipotent.

Proof.

Let Γ-seminearring R is a left duo and has a mate function f. Then a = aΓf(a)Γa for every a ∈ R. This gives a = af(a) af(a)Γa = $(af(a))^2$Γa. Since left duo R and a left ideal RΓa as such RΓaΓR ⊆ RΓa.

Then we can write (af(a))ΓaΓf(a) = bΓa for some element b in R, in which with a = $(af(a))^2$Γa makes a = bΓa^2 . This gives RΓa = RΓa^2 and therefore R is a left bipotent.

Proposition 3.2.

Any homomorphic image of a left duo Γ-seminearring R is a left duo Γ-seminearring.

Proof.

Let g : R → R' is a Γ-seminearring epimorphism and assume that a, r' ∈ R' . The fact that g is onto then there exist x, r ∈ R for which a = g(x), r' = g(r). Now

r'Γa= g(r)Γg(x)

= g(rΓx) (since rΓx ∈ RΓx = xΓR ⇒ rΓx = xΓr_1 for some r_1 ∈ R)

= g(x Γr_1) = g(x) Γg(r_1) = a Γr'_1 So R'Γa ⊂ aΓR'.

Similarly aΓR’ ⊂ RΓ’a. Thus R’Γa = aΓR’.

We obtained the following as a consequence of Proposition 3.2.

Theorem 3.3. [6, 7]

Each Γ-seminearring is isomorphic to a subdirect product of subdirectly irreducible Γ-seminearrings.

Theorem 3.4.

Every left duo Γ-seminearring R with mate functions is isomorphic to a subdirect product of subdirectly irreducible left duo Γ-seminearrings.

Proof.

By Theorem 3.3, R is mapping (isomorphic) to a subdirectly irreducible Γ-seminearrings subdirect product say R_i’s then R has homomorphic image of every R_i. The required outcomes are now obtained from Proposition 3.2.

Theorem 3.5.

If I is an ideal of a left duo Γ-seminearring, then R/I is also a left duo Γ-seminearring.

Proof.

We observe that R/I is a homomorphic representation of R under the canonical homomorphism then the required result follows directly from Proposition 3.2.

Theorem 3.6.

Let Boolean seminearring R be a left duo Γ-seminearring. Then R has no non-zero divisors if and only if R is simple.

Proof.

Obviously, as R is assume to be Boolean, then the identity map gives a mate function of R. For every x ∈ R, RΓx = RΓx^2 and if R has no non-zero divisors, it follows that R = RΓx for all x ∈ R-{0}. So R does not have any non-trivial ideals and therefore R is simple.

For the converse, let b ∈ R such that bΓa = 0 for some a ∈ R. Since R has left duo and simple RΓa = R. Therefore some y exists in R so it is b = yΓa. This yields 0 = bΓa =

$y\Gamma a^2 = y\Gamma a = b$ (as R is Boolean) and this implies b = 0. Therefore, the Γ-seminearring R contains no non-zero divisors.

Proposition 3.7

If R is a Γ-seminearring which has f then Γ-seminearring R is to be a left normal (right normal) Γ-seminearring.

Proof.

From our hypothesis, f is a mate of a Γ-seminearring R, ∀ t ∈ R, t = tΓf(t)Γt ∈ RΓt. Then it is obvious to say R is a left normal Γ-seminearring. Similarly R is right normal. Thus the result follows.

Proposition 3.8.

In R is a left duo, we get eΓf Γe = eΓf, eΓf ∈ E for e and f are idempotents.

Proof.

Suppose x ∈ R, then the left ideal of R is written as RΓx. The fact that Γ-seminearring R is a left duo, then RΓx is a right ideal of a seminearring R. Therefore, we have RΓxΓR ⊆ RΓx. It follows eΓf = eΓeΓf ∈ RΓeΓR ⊆ RΓe then eΓf = aΓe, a ∈ R. Therefore eΓfΓe = aΓe = eΓf. Finally eΓf ∈ E.

Conclusion:

A left ideal is not the same as a right ideal, not even in Γ-nearring theory. This enhances the appeal of researching a Γ-seminearring. In this work, we obtain a number of useful results regarding left duo of Γ-seminearrings that do not meet the aforementioned criterion.

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