Pseudo Edge Monophonic Number and Perfect Edge Monophonic Graph

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ABSTRACT

A set *M* of vertices of a connected graph *G* is a monophonic set if every vertex of *G* lies on an *x-y* monophonic path for some elements *x* and *y* in *M*. The minimum cardinality of a monophonic set of *G* is the monophonic number of *G*, denoted by $m(G)$. A set *M* of vertices of a graph *G* is an edge monophonic set if every edge of *G* lies on an *x-y* monophonic path for some elements *x* and *y* in *M*. The minimum cardinality of an edge monophonic set of *G* is the edge monophonic number of *G*, denoted by $m_1(G)$.A set of vertices M^1 in *G* is called Pseudo edge monophonic set if the set of vertices which are not belongs to any edge monophonic set of *G* and the maximum cardinality of a Pseudo edge monophonic set of *G* is its Pseudo edge monophonic number and is denoted by$m_1'(G)$. A Pseudo edge monophonic set of size m_1' (G) is said to be a m_1' set. Also if m_1' $(G) = 0$, then *G* is called perfect edge monophonic graph.

Keywords : Edge Monophonic number, Pseudo edge monophonic number, Perfect edge monophonic graph.

1.Introduction

By a graph $G = (V, E)$, we mean a simple graph of order atleast two. The order and size of *G* are denoted by *p* and *q* respectively. For basic graph theoretic terminology, we refer to Harary [5]. The neighborhood of a vertex *v* is the set $N(v)$ consisting of all vertices *u* which are adjacent with *v*. The closed neighborhood of a vertex *v* is the set $N[v] = N(v)\ U\{v\}$. A vertex *v* is an extreme vertex if the sub graph induced by its neighbors is complete. A vertex *v* is a semi-extreme vertex of *G* if the sub graph induced by its neighbors as a full degree vertex in $N(v)$. In particular, every extreme vertex is a semi - extreme vertex and a semi - extreme vertex need not be an extreme vertex.

For any set *M* of vertices of *G*, the induced subgraph $<M>$ is the maximal subgraph of *G* with vertex set M. A simplex of a graph *G* is a subgraph of *G* which is a complete graph. If $e = \{u, v\}$ is an edge of a graph *G* with $d(u) = 1$ and $d(v) > 1$, then we call *e* a pendent edge, *u*-leaf and *v* support vertex. Let $L(G)$ be the set of all leaves of a graph *G*.

For any two vertices *x* and *y* in a connected graph *G*, the distance $d(x, y)$ is the length of a shortest *x-y* path in *G*. An *x-y* path of length $d(x, y)$ is called an *x-y* geodesic. A vertex *v* is said to lie on an *x-y* geodesic *P* if *v* is a vertex of *P* including the vertices *x* and *y*. The eccentricity $e(v)$ of a vertex *v* in *G* is the maximum distance from *v* and *a* vertex of *G*. The minimum eccentricity among the vertices of *G* is the radius, rad (G) or $r(G)$ and the maximum eccentricity is its diameter, diam*G* of *G*.

The closed interval $I[x, y]$ consists of all vertices lying on some x-y geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a geodetic set if $I[S] = V$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic set of cardinality $g(G)$ is called *a g*-set. The geodetic number of a graph was introduced in[1, 6] and further studied in [2, 3, 4, 5]. A set S of vertices of a graph G is an edge geodetic set if every edge of G lies on an x-y geodesic for some elements x and y in S. The minimum cardinality of an edge geodetic set of G is the edge geodetic number of G denoted by $eg(G)$. The edge geodetic number was introduced and studied in [9].

A chord of a path $u_1, u_2, \ldots, u_k$ in G is an edge u_iu_j with $j \geq i + 2$. A u-v path P is called a monophonic path if it is a chordless path. A set M of vertices is a monophonic set if every vertex of G lies on a monophonic path joining some pair of vertices in$M$, and the minimum cardinality of a monophonic set is the monophonic number $m(G)$. A monophonic set of cardinality $m(G)$ is called an m - set of G. The monophonic number of a graph G was studied in [10]. A set M of vertices of a graph G is an edge monophonic set if every edge of G lies on an $x - y$ monophonic path for some elements x and y in M. The minimum cardinality of an edge monophonic set of G is the edge monophonic number of G, denoted by $em(G)$. The edge monophonic number of a graph was introduced and studied in [7]. Let G be a connected graph and M a minimum edge monophonic set of G. A subset $T \subseteq M$ is called a forcing subset for M is the unique minimum edge monophonic set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing edge monophonic number of

M, denoted by $f_1(M)$, is the cardinality of a minimum forcing subset of M. The forcing edge monophonic number of G, denoted by $f_1(G)$, is $f_1(G) = \min\{f_1(M)\}$,where the minimum is taken over all minimum edge monophonic sets M in G. The forcing edge monophonic number of a graph was introduced and studied in [8].

It is easily seen that a Pseudo edge monophonic set is not in general a complementary edge monophonic set in a graph G. Also the converse is not valid in general. We investigate those subsets of vertices of a graph that are not belongs to edge monophonic set. We call these sets pseudo edge monophonic sets

2. PSEUDO EDGE MONOPHONIC NUMBER

Definition 2.1 : Let $G = (V, E)$ be a connected graph and M be the edge monophonic set of G. Then the set of vertices which are not belongs to any edge monophonic set of G is the pseudo edge monophonic set M' of G and the maximum cardinality of M' is called Pseudo edge monophonic number and is denoted by $m_1'(G)$.

Example 2.2 : For the graph G given in figure 2.1, $M_1 = \{v_1, v_4, v_6\}$

$M_2 = \{v_2, v_3, v_6\}$ are the minimum edge monophonic sets of G and so $m_1(G) = 3$.

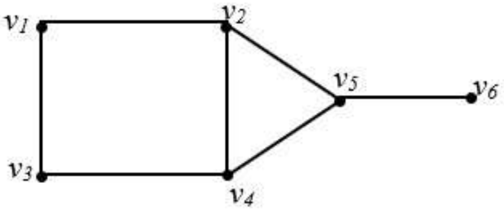

Figure 2.1: G

The pseudo edge monophonic set $M^1 = \{v_5\}$ and so $m_1'(G) = 1$

Remark 2.3 : For the graph G given in Figure 2.1, $M' = M^c$ and hence $m_1'(G) = |V\text{-}M|$. But in general $M^1$is not the complement set of M. For the graph G given in figure 2.1,
$M_1 = \{v_1, v_2, v_3, v_{6,}\}$ and $M_2 = \{v_1, v_2, v_4, v_6\}$ are the minimum two edge monophonic sets and so $m_1'(G) = 0$.

Theorem 2.4 : Let $M_1, M_2, \ldots, M_n$ are the minimum edge monophonic set of G, then
$m_1'(G) = \left|\bigcap_1^n M_i^c\right|$

Proof : Let M' be the pseudo edge monophonic set of G. It is enough to prove that $M' = \bigcap_1^n M_i^c$. Let $v \in V$ such that $v \in M'$. Then v doesn't belongs to any edge monophonic set of G.Hence $v \notin M_i \forall i (1 \leq i \leq n)$.

$\therefore v \in M_i^c \; \forall i \; (1 \leq i \leq n)$.

$\Rightarrow v \in \bigcap_1^n M_i^c$

$= M' \subseteq \bigcap_1^n M_i^c$

Let u be a vertex of G such that $u \in \bigcap_1^n M_i^c$

$u \in M_i^c \; \forall i \; (1 \leq i \leq n)$
$u \notin M_i \forall i (1 \leq i \leq n)$
Hence $u \in M'$ and so $M' = \bigcap_1^n M_i^c$

Theorem 2.5 : Let G = (p, q) be a connected graph. Then $m_1'(G) + m_1(G) = p$ if and only if $f_1(G) = 0$.

Proof : Suppose $m_1'(G) + m_1(G) = p$. Let M' be the Pseudo edge monophonic set and M be the minimum edge monophonic set of G. Then $M' \cup M = p$ and hence $M^1 = p - M$. Then by Theorem2.4, M is the unique minimum edge monophonic set of G and so the forcing edge monophonic number $f_1(G) = 0$.

Conversely, suppose that $f_1(G) = 0$, then G has the unique minimum edge monophonic set M. By the Theorem 2.4, $M' = M^c$ and hence $m_1'(G) + m_1(G) = p$.

Theorem 2.6 : For any connected Graph G, $0 \leq m_1'(G) \leq p - 2$.

Proof : An edge monophonic set needs atleast two vertices and therefore$m_1'(G) \leq p - 2$. Clearly, the set of all vertices of K_p is the edge monophonic set of G so that $m_1'(G) = 0$. Then $0 \leq m_1'(G) \leq p - 2$.

Remark 2.7 : The bounds in Theorem 2.6 are sharp. For the complete graph $Kp(p \geq 2)$, $m_1'(K_p) = 0$. The set of all non end vertices of a path P_p $(p \geq 2)$ is its Pseudo edge monophonic set so that $m_1'(P_p) = p - 2$. Thus the path P_p has the largest possible Pseudo edge monophonic number $p - 2$ and that the complete graph has the least Pseudo edge monophonic number 0.

Theorem 2.8 : No extreme vertex, in particular no end vertex belongs to Pseudo edge monophonic set.

Proof : Let M be an edge monophonic set of G and v be an extreme vertex of G. Let

$\{v_1, v_2, \ldots, v_k\}$ be the neighbors of v and $vv_i (1 \leq i \leq k)$ be the edges incident on v. since v is an extreme vertex, $v_i$and v_j are adjacent for $i \neq j$ $(1 \leq i, j \leq k)$ so that any monophonic path which contains vv_i $(1 \leq i \leq k)$ is either$v_i v$ or $u_1 u_2 \ldots u_1 v_i\, v$, where each $u_i (1 \leq i \leq l)$ is different from v_i. Hence it follows that $v \in M$ and hence $v \notin M'$.

Theorem 2.9 : For any connected graph G, every cut vertex of G belongs to Pseudo edge monophonic set of G.

Proof : Let M, M' be any minimum edge monophonic set, Pseudo edge monophonic set of G respectively. Let $v \in M$ be any vertex. We claim that v is not a cut vertex of G. Suppose that v is a cut vertex of G. Let $G_1, G_2, \ldots G_r$ $(r \geq 2)$ be the components of $G- v$. Then v is adjacent to aleast one vertex of G_i forevery i $(1 \leq i \leq r)$. Let $M_1 = M - \{v\}$. Let uw be an edge of G which lies on a monophonic path P joining a pair of vertices say x and v of M. Assume without loss of generality that $x \in G_1$. Since v is adjacent to at least one vertex of each G_i $(1 \leq i \leq r)$, assume that v is adjacent to a vertex y in $G_k (k \neq 1)$. Since M is an edge monophonic set, vy lies in a monophonicpath Q joining v and a vertex z of M such that z must necessarily belong to G_k. Thus $z \neq v$. Now, since v is a cut vertex of G, the union P U Q of the two monophonic paths P and Q is obviously a monophonic path in G joining x and z in M and thus the edge uw lies on this monophonic path joining the two vertices x and z of M_1. Thus we have proved that every edge that lies on a monophonic path joining a pair of vertices x and v of M also lies on a monophonic path joining two vertices of M_1. Hence it follows that every edge of G lies on a monophonic path joining two vertices of M_1, which shows that M_1 is an edge monophonic set of G. Since $|M_1| = |M_1| - 1$, this contradicts the fact that M is an edge monophonic set of G. Hence $v \notin M$ so that $v \in M'$.

Corollary 2.10 : For any connected graph G with k cut vertices, $k \leq m_1'(G) \leq p-2$.

Proof : This follows from Theorem 2.6 and 2.9.

Corollary 2.11 : For any non-trivial tree T, the Pseudo edge monophonic number $m_1'(T)$ equals the number of non end-vertices in T. In fact, the set of all non end-vertices of T is the Pseudo edge monophonic set of G.

Proof : This follows from Theorem 2.8 and 2.9.

Corollary 2.12 : For the complete graph K_p $(p \geq 2)$, $m_1'(K_p) = 0$.

Proof : Since every vertex of the complete graph $K_p (p \geq 2)$ is an extreme vertex, by the Theorem 2.8 $m_1'(K_p) = 0$.

Theorem 2.13 : If G has exactly one vertex v of degree $p-1$, then $m_1'(G) = 1$.

Proof : If G has exactly one vertex v of degree $p-1$, then $m_1'(G) = p-1$ and G has a unique minimum edge monophonic set consisting of all the vertices of G other than v. Then by the Theorem 2.5 $m_1'(G)=1$

Theorem 2.14 : For the complete bipartite graph $G = K_{m,n}$

(i) $m_1'(G) = 1$ if $n \geq 2, m = l$.

(ii) $m_1'(G) = max \{m, n\}$ if $m, n \geq 2$.

(iii) $m_1'(G) = 0$ if $m = n$

Proof : (i) This follows from Corollary 2.13.

(ii) Let $m, n \geq 2$. First assume that $m < n$.

Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of G. Let $M = U$. We prove that $M$is aminimum edge monophonic set of G.Any edge u_iw_j $(1 \leq i \leq m)$ and $(1 \leq j \leq n)$ lies on the monophonic path $u_iw_ju_k$ for any $k \neq i$ so that M is an edge monophonic set of G. Let T be any set of vertices such that $|T| < |M|$. If $T \subseteq U$, then there exists a vertex $u_i \in U$ such that $u_i \notin T$. Then for any edge $u_iw_j (1 \leq j \leq n)$, the only monophonic path containing u_iw_j are u_i,w_j,u_k ($k \neq i$) and w_j,u_i,w_l ($l \neq j$) and so u_iw_j cannot lie in a monophonic path joining two vertices of T. Thus T is not an edge monophonic set of G. If $T \subseteq W$, again T is not an edge monophonic set of G by a similar argument. If $T \subseteq U \cup W$ such that T contains at least one vertex from each of U and W, then, since $|T| < |M|$, there exist vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin T$ and $w_j \notin T$. Then clearly the edge u_iw_j does not lie on a monophonic path connecting two vertices of T so that T is not an edge monophonic set. Thus in any case T is not an edge monophonic set of G. Hence M is a minimum edge monophonic set so that $m_1(G) = |M| = m$. Now, if $m = n$, we can prove similarly that $M = U$ or W is a minimum edge monophonic set of G. Thus G has unique edge monophonic set U. Then by Theorem 2.5, $M' = W$ and hence $m_1^1(G) = n = max\{m,n\}$.

(iii) Suppose $m = n$. Then as in (ii) $U = \{u_1, u_2, ..., u_m\}$, $W = \{w_1, w_2, ..., w_m\}$ are the only edge monophonic set of G. Hence by the Theorem 2.4 $m_1^1(G) = 0$.

3. PERFECT EDGE MONOPHONIC GRAPH

Definition 3.1 : A connected graph G is said to be perfect edge monophonic graph if every vertex of G lies in anyone of the edge monophonic set of G. That is, G is perfect edge monophonic graph if $m_1'(G) = 0$.

Example 3.2 : For the graph G given in Figure 3.1, $M_1 = \{v_1,v_3,v_4,v_6\}$ and $M_2 = \{v_1,v_2,v_5,v_6\}$ are two minimum edge monophonic sets and so $m_1'(G) = 0$.

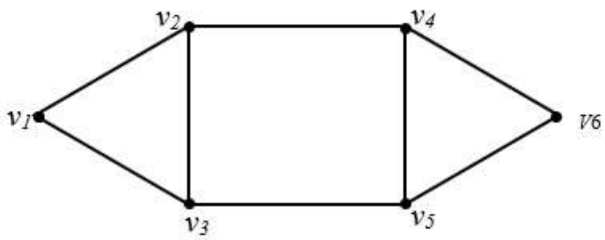

Figure 3.1 : G

Theorem 3.3 : If G has more than one vertex of degree $p-1$, then G is perfect edge monophonic graph.

Proof : If all the vertices are of degree $p-1$, then $G = K_p$, then by the **Corollary 2.12**, $m_1'(G) = 0$ and hence G is perfect edge monophonic graph. Otherwise, let $v_1, v_2, \ldots, v_k (2 \leq k \leq p-2)$ be the vertices of degree $p-1$. Suppose $m_1'(G) > 0$, then $m_1(G) < p$. Let M be a minimum edge monophonic set of G such that $|M| < p$. Then M contains all the vertices $v_1, v_2, \ldots, v_k$. Let v be a vertex such that $v \notin M$. Then $deg(v) < p - 1$. Since any two of $v_1, v_2, \ldots, v_k$ are adjacent, the edge $vv_i (1 \leq i \leq k)$ cannot lie on a monophonic path joining a pair of vertices v_j and v_l ($j \neq l$). Similarly, since any v_j is adjacent to any vertex of M, which is different from $v_1, v_2, \ldots, v_k$, the edge $vv_i (1 \leq i \leq k)$ cannot lie on a monophonic path joining a vertex v_j and a vertex of M, which is different from $v_1, v_2, \ldots, v_k$. Now, let u and w be vertices of M different from $v_1, v_2, \ldots, v_k$. Since v_i is adjacent to both u and w and $d(u, v) \leq 2$, the edge vv_i cannot lie on a monophonic path joining u and w. Thus we see that the edges $vv_i (1 \leq i \leq k)$ do not lie on any monophonic path joining a pair of vertices of M, which is a contradiction to the fact that M is a minimum edge monophonic set of G. Hence $m_1'(G) = 0$. Then G is perfect edge monophonic set of G.

Theorem 3.4 : Any complete bipartite graph $G = K_{m,n}$ is perfect if and only if m = n.
Proof : Suppose $G = K_{m,n}$ is perfect. Then by definition $m_1'(G) = 0$. Suppose $m < n$. Then by the Theorem 2.14 (ii) $m_1'(G) = n$, is a contradiction to the hypothesis. Hence $m = n$. Conversely suppose $m = n$, then by the Theorem 2.14 (iii) $m_1'(G) = 0$, therefore $G = K_{m,n}$ is perfect.

Theorem 3.5 : Any complete graph $K_p (p \geq 2)$ is perfect.
Proof : The proof follows corollary 2.12.
Theorem 3.6 : Any even cycle C_p is perfect edge monophonic graph.
Proof : Since C_p is an even cycle, $m_1(G) = 2$.
Also $M_1 = \{v_1, v_{\frac{p}{2}+1}\}$, $M_2 = \{v_2, v_{\frac{p}{2}+2}\}$ …, $M_{\frac{p}{2}} = \{v_{\frac{p}{2}}, v_p\}$ are the only minimum edge monophonic set of C_p. Then by the Theorem 2.4, $m_1'(G) = 0$ and hence C_p is perfect edge monophonic graph.

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