Using the One-Dimensional Geometric AH-Isometry, Neutrosophic Real Analysis is studied.

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Abstract

With many real analysis principles, such as continuity, differentiability, and integrability, this paper aims to analyse and define the neutrosophic real functions with one neutrosophic variable depending on the geometric isometry (AH-Isometry). Different common functions in the neutrosophic environment, such as the logarithmic function, exponential function, and trigonometric functions, have formal formulations that we have described. Rising to neutrosophic powers is clearly described as rising to neutrosophic numbers to any power.

Keywords: Integration, Differentiation, Continuity, AH-isometry, Neutrosophic Real Analysis.

Introduction

Neutrosophy is a new branch of philosophy concerns with the indeterminacy in all areas of life and science. Ithas become a useful tool in generalizing many classical systems such as equations [30], number theory , topology, linear spaces, modules , and ring of matrices [1-7,12-21].

In the literature, we find many studies about neutrosophic calculus, where some definitions and properties were presented about neutrosophic real functions and numbers [9-11,23-33].

The neutrosophic real functions with one variable were defined only in a special case, as follows:

$(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}) + \boldsymbol{h}(\boldsymbol{x})\boldsymbol{I}$ where $\boldsymbol{I}$ takes an interval value defining what is called by neutrosophic thick functions .For example$(\boldsymbol{x}) = \boldsymbol{2x} + \boldsymbol{5xI}, \boldsymbol{I} \in [\boldsymbol{0}, \boldsymbol{0}.\boldsymbol{01}]$ is a neutrosophic real thick function.

The problem with this definition, that it does not consider the general case $\boldsymbol{f}: (\boldsymbol{I}) \to \boldsymbol{R}(\boldsymbol{I}); \boldsymbol{f} = \boldsymbol{f}(\boldsymbol{X})$ and $\boldsymbol{X} = \boldsymbol{x} + \boldsymbol{yI} \in \boldsymbol{R}(\boldsymbol{I})$.

Recently, Abobala et.al, have presented the concept of two-dimensional AH-isometry to study thecorrespondence between neutrosophic plane$(\boldsymbol{I}) \times \boldsymbol{R}(\boldsymbol{I})$ and the classical module $\boldsymbol{R^2} \times \boldsymbol{R^2}$. Also, the one-dimensional AH-isometry between R(I) and $\boldsymbol{R} \times \boldsymbol{R}$. This isometry was useful in defining inner products and norms , ordering , and neutrosophic geometrical shapes.

In this work, we use the one-dimensional AH-isometry to turn the general case of neutrosophic real functions with one variable into two classical real functions so we will go from $(\boldsymbol{I})$ space into $\boldsymbol{R} \times \boldsymbol{R}$ space, we study the properties of our functions then we go back to $\boldsymbol{R}(\boldsymbol{I})$ space using AH-isometry.

This work will provide for the first time an algorithm to compute the neutrosophic powers of neutrosophic numbers including neutrosophic powers which wasn't studied before, we will present and define neutrosophic continuity, differentiation, integration and lots of popular neutrosophic functions like neutrosophic exponential function, neutrosophic logarithmic function and neutrosophic trigonometric functions.

Definitions and theorems presented in this paper are very useful to define mathematically lots of concepts including differential equations, integral equations, probability distribution functions,... etc.

1. Neutrosophic Functions on $(\boldsymbol{I})$

Definition 2.1

Let $(I) = \{a + bI\ ; a, b \in R\}$ where $I^2 = I$be the neutrosophic field of reals. The one-dimensionalisometry (AH-Isometry) is defined as follows: [49]

$$T: R(I) \to R \times R$$
$$T(a + bI) = (a, a + b)$$

Remark:

T is an algebraic isomorphism between two rings, it has the following properties:

1) T is bijective.
2) T preserves addition and multiplication, i.e.:

$$T[(a+bI)+(c+dI)] = T(a+bI)+T(c+dI)$$

And

$$T[(a+bI)\cdot(c+dI)] = T(a+bI)\cdot T(c+dI)$$

3) Since T is bijective, then it is invertible by:

$$T^{-1}: R\times R \to (I)$$
$$T^{-1}(a,b) = a+(b-a)I$$

4) T preserves distances, i.e.:

The distance on $R(I)$ can be defined as follows:

Let $A = a+bI, B = \;\; + dI$ be two neutrosophic real numbers, then $L = \|\vec{AB}\| = d[(a+bI, c+dI)] = |a+bI-(c+dI)| = |(a-c)+I(b-d)| = |a-c| + I[|a+b-c-d| - |a-c|]$.

On the other hand, we have:

$T(\|\vec{AB}\|) = (|a-c|, |(a+b)-(c+d)|) = (d(a,c), d(a+b,c+d)) = d[(a,a+b),(c,c+d)]$=$d(T(a+bI),T(c+dI))$

=$\|T(\vec{AB})\|$.

This implies that the distance is preserved up to isometry. i.e.$\|(AB)\| = T(\|AB\|)$

Definition 2.2

Let $f:(I)\to R(I); f = f(X)$ and $X = x+yI \in R(I)$ the f is called a neutrosophic real function with one neutrosophic variable.

Example:

Take $f:(I)\to R(I); f(X) = X^2 + IX + 2I = (x+yI)^2 + I(x+yI) + 2I = x^2 + (y^2 + 2xy + x + y + 2)$

In the following we are going to show how to study analytical properties of the function defined in definition 2.2

Remark:

Using the one-dimensional AH-isometry we can turn any neutrosophic real function into two classical real functions, i.e., to the classical Euclidean plane $R\times R$.

Example:

Consider the following neutrosophic real function

$f:(I)\to R(I) : f(X) = IX - 1 + 3I$,

We can write: $(f(X)) = T(I)T(X) + T(-1+3I)$
$= (0,1)(x, x+y) + (-1,2)$,
$= (-1, x+y+2)$,

Assuming that $X = 1+I$, i.e., $x = y = 1$ we get:

$(f(1+I)) = (-1,4) \Rightarrow f(1+I) = T^{-1}(-1,4) = -1+5I$,By direct computing we can find:

$(1+I) = I(1+I) - 1 + 3I = -1+5I$,

Theorem 2.1

Let $f:(I)\to R(I)$ be a neutrosophic real function with one variable,$X = x+yI \in R(I)$ then f can be turnedinto two classical real functions.

Proof

Since the direct image of the variable X by the one-dimensional AH-isometry is $(X) = (x, x+y)$ we get that $(f(X)) = (h(x), g(x+y))$ where: $h, g: R\to R$ are classical real functions.

Definition 2.3

Let $f:(I)\to R(I)$ be a neutrosophic real function with one variable. We say that f is integrable, continuous, or differentiable on $]+bI, c+dI[$ iff $T(f(X))$ is integrable, continuous, or differentiable on $T(\,]a+bI, c+dI[\;\;)$

Example:

Let ( ) ( ) ( ) 1 2 we have:

$f: RI \to R\ I\ ; f\ X =_2 X + I$

1) $1 + I \leq 1 + 2I$ because $1 \leq 1, 1 + 1 \leq 1 + 2$
2) f is integrable on $[1 + I, 1 + 2I]$ that is because

$$(f(X)) = T\left(\frac{1}{2}\right) T(X^2) + T(I),$$

$$= \left(\frac{1}{2}, \frac{1}{2}\right)(x^2, (x + y)^2) + (0,1),$$

$$= \left(\frac{1}{2}x^2, \frac{1}{2}(x + y)^2 + 1\right),$$

$$([1 + I, 1 + 2I]) = [(1,2), (1,3)] = ([1,1], [2,3]),$$

3) Now, we integrate f as following:

a) by using T:

We have:

$$\int_1^1 \frac{1}{2}x^2\, dx = 0,$$

Also:

$$\int_2^3 \left[\frac{1}{2}(x + y)^2 + 1\right] d(x + y),$$

$$= \left[\frac{1}{6}(x + y)^3 + (x + y)\right]_2^3$$

$$= \frac{1}{6}27 + 3 - \frac{1}{6}8 - 2 = \frac{25}{6},$$

So:

$$T^{-1}\left(0, \frac{25}{6}\right) = \frac{25}{6}I = \int_{1+I}^{1+2I} (X)dX,$$

b) by direct computing:

$$\int_{1+I}^{1+2I} \left(\frac{1}{2}{}^2 \quad + I\right) dX = \left[\frac{1}{6}X^3 + IX\right]_{1+I}^{1+2I},$$

$$= \frac{1}{6}(1 + 2I)^3 + I(1 + 2I) - \frac{1}{6}(1 + I)^3 - I(1 + I),$$

$$= \frac{1}{6}(1 + 6I + 12I + 8I) + 3I - \frac{1}{6}(1 + 3I + 3I + I),$$

$$= \frac{25}{6}I,$$

Notice that we get the same result!

2. Computing Powers in $R(I)$

To compute such equation: $(a + bI)^{+dI}$; $a, b, c, d \in R$ we need the one-dimensional isometry again:

$[(a + bI)^{c+dI}] = (a, a + b)^{(c,c+d)} = (a^c, (a + b)^{c+d}),$

Which results:

$(a + bI)^{c+dI} = T^{-1}(a^c, (a + b)^{c+d}),$

$= a^c + I[(a + b)^{c+d} - a^c].$

Example:

let $A = (2 + I)^I$, we have:

$(A) = (2,3)^{(0,1)} = (2^0, 3^1) = (1,3).$

$\Rightarrow A = T^{-1}(1,3) = 1 + 2I.$

3. Neutrosophic Trigonometric Functions:

In the following theorem, we are going to represent and provide formulas of trigonometric functions with neutrosophic angle of the form $\theta_N = a + bI$ and some properties according to it.

Theorem 4.1:

Let (I) be the neutrosophic field of reals, we have:

1. $\sin(a + bI) = \sin a + I[\sin(a + b) - \sin a\ ]$
2. $\cos(a + bI) = \cos a + I[\cos(a + b) - \cos a]$
3. $\tan(a + bI) = \tan a + I[\tan(a + b) - \tan a]$
4. $\sin^2(a + bI) + \cos^2(a + bI) = 1$

Proof:

1. $\sin(a+bI) = \frac{e^{a+bI}-e^{-(a+bI)}}{2i} = A$, $i^2 = -1$.

$$T(A) = T\left(\frac{1}{2i}\right) T\left(e^{a+bI} - e^{-(a+bI)}\right)$$
$$= \left(\frac{1}{2i}\right)\left[(e,e)^{(a,a+b)} - (e,e)^{(-a,-a-b)}\right]$$
$$= \frac{1}{2i}\left[(e^a, e^{a+b}) - (e^{-a}, e^{-a-b})\right]$$
$$\Rightarrow A = \frac{1}{2i}\left(e^a + (e^{a+b} - e^a) - e^{-a} - I(e^{-a-b} - e^{-a})\right)$$
$$= \frac{e^a - e^{-a}}{2i} + I\left[\frac{e^{a+b} - e^{-a-b}}{2i} - \frac{e^a - e^{-a}}{2i}\right]$$
$$= \sin a + [\sin(a+b) - \sin a]$$

Notice that we can compute $\sin(a+bI)$directly as follows:

$(\sin(a+bI)) = \sin[T(a+bI)] = \sin(a, a+b) = (\sin a, \sin(a+b))$,

$\Rightarrow \sin(a+bI) = \sin a + I[\sin(a+b) - \sin a]$,

2. similar to proof 1
3. similar to proof 1
4. $\sin^2(a+bI) = (\sin a + I[\sin(a+b) - \sin a])^2$

$= \sin^2 a + 2I \sin a \sin(a+b) - 2I \sin^2 a + I\,(\sin^2(a+b) - 2\sin(a+b)\sin a + \sin^2 a)$,

$= \sin^2 a + [2\sin a \sin(a+b) - 2\sin^2 a + \sin^2(a+b) - 2\sin(a+b)\sin a + \sin^2 a]$,

$= \sin^2 a + [\sin^2(a+b) - \sin^2 a]$,Similarly,

we find that:

$\cos^2(a+bI) = \cos^2 a + I[\cos^2(a+b) - \cos^2 a]$,

So:

$\sin^2(a+bI) + \cos^2(a+bI) = \sin^2 a + \cos^2 a + I[\sin^2(a+b) - \sin^2 a + \cos^2(a+b) - \cos^2 a]$,

$= 1 + [1-1]$,

$= 1$.

Theorem 4.2:

1. $-1 \le \sin(a+bI) \le 1$
2. $-1 \le \cos(a+bI) \le 1$

Proof:

1. We have $(-1,-1) \le [\sin(a+bI)] = (sina, \sin(a+b)) \le (1,1)$, thus

$T^{-1}(-1,-1) \le \sin(a+bI) \le T^{-1}(1,1), i.e. -1 \le \sin\,(a+bI) \le 1$.

2. The proof is similar to 1.

Remarks:

1. $\sin I\pi = \sin 0 + I[\sin\pi - \sin 0] = 0$
2. $\cos I\pi = \cos 0 + I[\cos\pi - \cos 0] = 1 + I[-1-1] = 1 - 2I$
3. $\sin\frac{\pi}{2}I = \sin 0 + I\,[\sin\frac{\pi}{2} - \sin 0] = I$
4. $\cos\frac{\pi}{2}I = \cos 0 + I\,[\cos\frac{\pi}{2} - \cos 0] = 1 - I$
5. $\sin\frac{\pi}{4}I = \frac{1}{\sqrt{2}}I$
6. $\cos\frac{\pi}{4}I = 1 + (\frac{1}{\sqrt{2}} - 1)\,I$
7. $\tan\frac{\pi}{4}I = I = \frac{\frac{1}{\sqrt{2}}I}{1+(\frac{1}{\sqrt{2}}-1)}$

5. Neutrosophic exponential and logarithmic functions:

In this section we are going to define and find the mathematical form of the function $(X) = e^X$ where $X = x + Iy$:

Theorem 5.1

Let (I) be the neutrosophic field of reals, we have:

1. $e^{x+Iy} = e^x + (e^{x+y} - e^x)$
2. $\ln(x+Iy) = \ln x + I(\ln(x+y) - \ln(x))$, where $x + yI > 0$.

Proof:

1. Let $A = e^X$:

$(A) = T(e^X) = T(e^{x+Iy}) = e^{T(x+Iy)} = e^{(x,x+y)} = (e^x, e^{x+y}) \Rightarrow A = T^{-1}(e^x, e^{x+y}) = e^x + I(e^{x+y} - e^x)$.

2. Let:

$$a_1 + a_2 I = \ln X$$

$$\Rightarrow X = x + Iy = e^{a_1 + a_2 I} = e^{a_1} + (e^{a_1 + a_2} - e^{a_1})$$
$$\Rightarrow x = e^{a_1}, y = e^{a_1 + a_2} - e^{a_1}$$
$$\Rightarrow a_1 = \ln(x), y = x \cdot e^{a_2} - x$$
$$\Rightarrow \frac{y}{x} + 1 = e^{a_2}$$
$$\Rightarrow a_2 = \ln\left(\frac{x + y}{x}\right) = \ln(x + y) - \ln(x)$$

Which yields:

$$\ln(x + Iy) = \ln x + I(\ln(x + y) - \ln(x))$$

Example 5.1:

Let's calculate $e^I, \ln(1 + I)$:

$$e^I = e^{0+I} = e^0 + (e - e^0) = 1 + (e - 1)I$$
$$\ln(1 + I) = \ln(1) + I(\ln(2) - \ln(1)) = \ln(2)\, I$$

It is easy to check that the properties of exponential/logarithmic function are still true in the neutrosophic case.

6. Conclusions

In this paper we have studied some concepts of neutrosophic real analysis depending on the one-dimensional AH-isometry. We have provided a strict definition of continuity, integration and differentiation of a neutrosophic functions with neutrosophic variables. We have presented an algorithm to compute powers of neutrosophic numbers to neutrosophic powers. Definitions of trigonometric functions, exponential functions and logarithmic functions were presented and lots of their properties had been proved.

Future research directions

Content of this work is very helpful in defining and solving many open problems in neutrosophic theory including differential equations, integral equations and probability distribution functions. We aim to do a lots of works in these branches in the future.

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