A cotangent function-related half-discrete Hilbert inequality in the entire plane with the constant factor

Mr. JYOTI RANJAN PANDA,
NALANDA INSTITUTE OF TECHNOLOGY,BHUBANESWAR
EMAIL-jyotiranjan@thenalanda.com

Abstract

1 A novel half-discrete kernel function in the entire plane is defined in this work by the addition of a few parameters, and it takes into account both the homogeneous and non-homogeneous instances. A novel half-discrete Hilbert-type inequality with the new kernel function and its equivalent Hardy-type inequalities are established by using some actual analysis approaches, particularly the manner of the weight function. Furthermore, it is demonstrated that the constant factors of the recently discovered inequalities are optimal. Finally, novel half-discrete Hilbert-type inequalities with unique kernels are introduced at the conclusion of the work by giving particular values to the parameters.

Key terms and phrases: homogeneous kernel; non-homogeneous kernel; half-discrete; Hilbert-type inequality; Hardy-type inequality.

2 Introduction

Suppose that $p > 1$, Θ is a measurable set, and $f(x), \mu(x)$ are two non-negative measurable functions defined on Θ. Define

$$L_{p,\mu}(\Theta) := \left\{ f : \|f\|_{p,\mu} := \left(\int_{\Theta} f^p(x)\mu(x)\mathrm{d}x \right)^{1/p} < \infty \right\}.$$

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Specially, if $\mu(x) \equiv 1$, then we have the abbreviations: $\|f\|_p := \|f\|_{p,\mu}$, and $L_p(\Theta) := L_{p,\mu}(\Theta)$.

In addition, let $p > 1$, $a_n, v_n > 0$, $n \in \Pi \subseteq \mathbb{Z}$, $a = \{a_n\}_{n\in\Pi}$. Define

$$l_{p,v} := \left\{ a : \|a\|_{p,v} := \left(\sum_{n\in\Pi} a_n^p v_n \right)^{1/p} < \infty \right\}.$$

Specially, if $v_n \equiv 1$, then we have the abbreviations: $\|a\|_p := \|a\|_{p,v}$, and $l_p := l_{p,v}$.

Consider two real-valued sequences: $a = \{a_m\}_{m=1}^{\infty} \in l_2$, and $b = \{b_n\}_{n=1}^{\infty} \in l_2$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \pi \|a\|_2 \|b\|_2, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) was proposed by D. Hilbert in his lectures on integral equations in 1908, and Schur established the integral analogy of (1.1) in 1911, that is,

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x+y} dxdy < \pi \|f\|_2 \|g\|_2, \tag{1.2}$$

where $f, g \in L_2(\mathbb{R}^+)$, and the constant factor π is also the best possible.

Inequalities (1.1) and (1.2) are commonly named as Hilbert inequality [1]. In recent decades, especially after the 1990s, a great many extended forms of (1.1) and (1.2) were established, such as the following one provided by M. Krnić and J. Pečarić [2]:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|a\|_{p,\mu} \|b\|_{q,\nu}, \tag{1.3}$$

where $0 < \lambda \leq 4$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu_m = m^{p(1-\lambda/2)-1}$, $\nu_n = n^{q(1-\lambda/2)-1}$, and $B(u, v)$ is the Beta function [3].

Moreover, Yang [4] established the following extension of (1.2), that is,

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x^{\lambda}+y^{\lambda}} dxdy < \frac{\pi}{\lambda \sin \beta\pi} \|f\|_{p,\mu} \|g\|_{q,\nu}, \tag{1.4}$$

where $\beta, \gamma, \lambda > 0$, $\beta + \gamma = 1$, $\mu(x) = x^{p(1-\lambda\beta)-1}$, and $\nu(x) = x^{q(1-\lambda\gamma)-1}$.

With regard to some other extensions of inequalities (1.1) and (1.2), we refer to [5ś11]. Such inequalities as (1.3) and (1.4) are commonly known as Hilbert-type inequality. It should be pointed out that, by introducing new kernel functions, and considering the coefficient refinement, reverse form, multi-dimensional extension, a large number of Hilbert-type inequalities were established in the past 20 years (see [12ś23]).

It should also be pointed out that the kernel function in inequalities (1.1) and (1.2) are homogeneous [11, 12], and there exists another form of (1.1) with a non-homogeneous kernel function [12], that is,

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1+xy}\,dxdy < \pi\|f\|_2\|g\|_2. \tag{1.5}$$

The discrete form of (1.5) can also be established, but its constant factor can not be proved to be the best possible (see [12], p. 315). In 2005, Yang provided a half-discrete form of (1.5) and proved that constant factor is the best possible, that is [24],

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{1+nx}\,dx < \pi\|f\|_2\|a\|_2. \tag{1.6}$$

With regard to some other half-discrete inequalities with homogeneous and non-homogeneous kernels, we refer to [23, 25ś32].

The main objective of this work is to establish a new class of half-discrete Hilbert- type inequalities defined in the whole plane with the kernel functions involving both the homogeneous and non-homogeneous cases, such as the following two:

$$\int_{-\infty}^\infty f(x)\sum_{n\in\mathbb{Z}^0} \frac{a_n}{1+(xn)^\beta+(xn)^{2\beta}}\,dx < \frac{2\sqrt{3}\pi}{3\beta}\|f\|_{p,\mu}\|a\|_{q,\nu}, \tag{1.7}$$

$$\int_{-\infty}^\infty f(x)\sum_{n\in\mathbb{Z}^0} \frac{a_n}{x^{2\beta}-(xn)^\beta+n^{2\beta}}\,dx < \frac{(4+\sqrt{3})\pi}{3\beta}\|f\|_{p,\tilde{\mu}}\|a\|_{q,\tilde{\nu}}, \tag{1.8}$$

where $\mu(x) = |x|^{p(1-\beta)-1}$, $\nu_n = |n|^{q(1-\beta)-1}$, $\tilde{\mu}(x) = |x|^{p(1-3\beta/2)-1}$, and $\tilde{\nu}_n = |n|^{q(1-\beta/2)-1}$.

More generally, a new kernel function with multiple parameters, which unifies some homogeneous and non-homogeneous cases is constructed, and then a half-discrete Hilbert-type inequality and its equivalent forms defined in the whole plane are established. The paper is organized as follows: detailed lemmas will be presented in Section 2, and main results and some corollaries will be presented in Section 3 and Section 4, respectively.

3 some lemmas

Lemma 2.1. *Let* $\delta \in \{1, -1\}$, *and*

$$\Omega := \left\{ t : t = \frac{2i+1}{2j+1}, i, j \in \mathbb{Z} \right\}.$$

Suppose that $\alpha \in (0,1)$, $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, *and* α, β, γ *satisfy* $\beta < \gamma$ *and* $\alpha + \beta < 1$. *Define*

(2.1) $$K(z) := \frac{1 - \delta z^{\beta}}{1 - \delta z^{\gamma}},$$

where $z \neq 1$ *for* $\delta = 1$, *and* $z \neq -1$ *for* $\delta = -1$. *Let* $K(1) := \frac{\beta}{\gamma}$ *for* $\delta = 1$, *and* $K(-1) := \frac{\beta}{\gamma}$ *for* $\delta = -1$. *Then*

(2.2) $$G(z) := K(z)\,|z|^{\alpha-1}$$

decreases monotonically on $\mathbb{R}^+$, *and increases monotonically on* $\mathbb{R}^-$.

Proof. We first consider the case where $\delta = 1$, and $z \in (0,1) \cup (1,\infty)$, then we have

(2.3) $$\frac{dK}{dz} = (1 - z^{\gamma})^{-2} z^{\gamma-1} H(Z),$$

where

(2.4) $$H(z) = (\beta - \gamma) z^{\beta} - \beta z^{\beta-\gamma} + \gamma.$$

It can easily get that

(2.5) $$\frac{dH}{dz} = \beta(\beta - \gamma) z^{\beta-1} - \beta(\beta - \gamma) z^{\beta-\gamma-1} = \beta(\beta - \gamma) z^{\beta-\gamma-1}(z^{\gamma} - 1).$$

Therefore, we have $\frac{dH}{dz} > 0$ for $z \in (0,1)$, and $\frac{dH}{dz} < 0$ for $z \in (1,\infty)$. It follows that $H(z) \leq H(1) = 0$. By (2.3), we get $\frac{dK}{dz} < 0$ for $z \in (0,1) \cup (1,\infty)$, and therefore $K(z)$ decreases monotonically on $\mathbb{R}^+$ for $\delta = 1$. Since $0 < \alpha < 1$, it can also be obtained that $G(z) = K(z) z^{\alpha-1}$ decreases monotonically on $\mathbb{R}^+$ for $\delta = 1$.

Second, consider the case of $\delta = 1$, and $z \in (-\infty, 0)$. Setting $z = -u$, $u \in (0,\infty)$, and observing that $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, we obtain

(2.6) $$G(z) = \frac{1 - z^{\beta}}{1 - z^{\gamma}} |z|^{\alpha-1} = \frac{1 + u^{\beta}}{1 + u^{\gamma}} u^{\alpha-1} := L(u).$$

In view of that $0 < \alpha < 1$, and $\alpha + \beta < 1$, we get

(2.7) $$\frac{dL}{du} = -\, u^{\alpha-2}(1 + u^{\gamma})^{-2}\ (1 - \alpha - \beta) u^{\beta} + (1 - \alpha - \beta + \gamma) u^{\beta+\gamma} + (1 - \alpha + \gamma) u^{\gamma} + 1 - \alpha \ < 0.$$

It implies that $L(u)$ decreases monotonically with u ($u \in \mathbb{R}^+$), and therefore $G(z)$ increases monotonically with z ($z \in \mathbb{R}^-$).

Lemma 2.1 is proved for $\delta = 1$. Additionally, in view of $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, it follows from the above discussions that Lemma 2.1 holds obviously for the case where $\delta = -1$. □

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Lemma 2.2. *Let* $\delta \in \{1, -1\}$, *and*

$$\Omega := \left\{ t : t = \frac{2i+1}{2j+1}, i, j \in \mathbb{Z} \right\}$$

Suppose that $\alpha \in (0,1)$, $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, *and* α, β, γ *satisfy* $\alpha + \beta < \gamma$. *Let* $\Phi(z) = \cot z$, *and* $K(z)$ *be defined by* (2.1). *Then*

(2.8)
$$\int_{-\infty}^{\infty} K(z)\, |z|^{\alpha-1}\, dz = \frac{\pi}{\gamma}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right].$$

Proof. Consider the case where $\delta = 1$. Observing that $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, we get

(2.9)
$$\begin{aligned}
\int_{-\infty}^{\infty} K(z)\, |z|^{\alpha-1}\, dz &= \int_{-\infty}^{0} \frac{1-z^\beta}{1-z^\gamma} z^{\alpha-1} dz + \int_{0}^{\infty} \frac{1+z^\beta}{1+z^\gamma} z^{\alpha-1} dz \\
&= \int_{0}^{1} \frac{1-z^\beta}{1-z^\gamma} z^{\alpha-1} dz + \int_{1}^{\infty} \frac{1-z^\beta}{1-z^\gamma} z^{\alpha-1} dz \\
&\quad + \int_{0}^{1} \frac{1+z^\beta}{1+z^\gamma} z^{\alpha-1} dz + \int_{1}^{\infty} \frac{1+z^\beta}{1+z^\gamma} z^{\alpha-1} dz \\
&= \int_{0}^{1} \frac{z^{\alpha-1} - z^{\alpha+\beta-1}}{1-z^\gamma} dz + \int_{0}^{1} \frac{z^{\gamma-\alpha-\beta-1} - z^{\gamma-\alpha-1}}{1-z^\gamma} dz \\
&\quad + \int_{0}^{1} \frac{z^{\alpha-1} + z^{\alpha+\beta-1}}{1+z^\gamma} dz + \int_{0}^{1} \frac{z^{\gamma-\alpha-\beta-1} + z^{\gamma-\alpha-1}}{1+z^\gamma} dz \\
&= 2\left[\int_{0}^{1} \frac{z^{\alpha-1} - z^{2\gamma-\alpha-1}}{1-z^{2\gamma}} dz + \int_{0}^{1} \frac{z^{\gamma-\alpha-\beta-1} - z^{\alpha+\beta+\gamma-1}}{1-z^{2\gamma}} dz\right] \\
&:= 2(J_1 + J_2).
\end{aligned}$$

Expanding $\frac{1}{1-z^{2\gamma}} (z \in (0,1))$ into a power series, and using Lebesgue term-by-term integration theorem, we get

(2.10)
$$\begin{aligned}
J_1 &= \int_{0}^{1} \sum_{j=0}^{\infty} \left(z^{2\gamma j+\alpha-1} - z^{2\gamma j+2\gamma-\alpha-1}\right) dz \\
&= \sum_{j=0}^{\infty} \int_{0}^{1} \left(z^{2\gamma j+\alpha-1} - z^{2\gamma j+2\gamma-\alpha-1}\right) dz \\
&= \sum_{j=0}^{\infty} \left(\frac{1}{2\gamma j+\alpha} - \frac{1}{2\gamma j+2\gamma-\alpha}\right).
\end{aligned}$$

Observing that $\Phi(z) = \cot z$ $(0 < z < \pi)$ can be written as the following rational fraction expansion [3]:

$$\Phi(z) = \frac{1}{z} + \sum_{j=1}^{\infty} \left(\frac{1}{z+j\pi} + \frac{1}{z-j\pi}\right),$$

we get

(2.11)
$$
\begin{aligned}
\Phi\left(\frac{\alpha\pi}{2\gamma}\right) &= \frac{2\gamma}{\pi}\left[\frac{1}{\alpha}+\sum_{j=1}^{\infty}\left(\frac{1}{2\gamma j+\alpha}+\frac{1}{\alpha-2\gamma j}\right)\right]\\
&= \frac{2\gamma}{\pi}\lim_{n\to\infty}\left(\sum_{j=0}^{n}\frac{1}{2\gamma j+\alpha}+\sum_{j=1}^{n}\frac{1}{\alpha-2\gamma j}\right)\\
&= \frac{2\gamma}{\pi}\lim_{n\to\infty}\left(\sum_{j=0}^{n}\frac{1}{2\gamma j+\alpha}-\sum_{j=0}^{n-1}\frac{1}{2\gamma j+2\gamma-\alpha}\right)\\
&= \frac{2\gamma}{\pi}\lim_{n\to\infty}\left[\frac{1}{2\gamma n+2\gamma-\alpha}+\sum_{j=0}^{n}\left(\frac{1}{2\gamma j+\alpha}-\frac{1}{2\gamma j+2\gamma-\alpha}\right)\right]\\
&= \frac{2\gamma}{\pi}\sum_{j=0}^{\infty}\left(\frac{1}{2\gamma j+\alpha}-\frac{1}{2\gamma j+2\gamma-\alpha}\right).
\end{aligned}
$$

Combining (2.10) and (2.11), we get

(2.12)
$$J_1 = \frac{\pi}{2\gamma}\Phi\left(\frac{\alpha\pi}{2\gamma}\right).$$

Similarly, we have

(2.13)
$$J_2 = -\frac{\pi}{2\gamma}\Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right).$$

Plugging (2.12) and (2.13) into (2.9), we arrive at (2.8) for $\delta = 1$. Additionally, if $\delta = -1$, then it is obvious that (2.9) is still valid owing to $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, and it follows therefore that (2.8) holds for the case where $\delta = -1$. Lemma 2.2 is proved. □

Lemma 2.3. *Let* $\delta \in \{1, -1\}$, *and*

$$\Omega := \left\{t : t = \frac{2i+1}{2j+1}, i, j \in \mathbb{Z}\right\}.$$

Suppose that $\alpha \in (0, 1)$, $\tau \in \Omega$, $\kappa \in (0, 1] \cap \Omega$, $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, *and* α, β, γ *satisfy* $\beta < \gamma$ *and* $\alpha + \beta < \min\{1, \gamma\}$. *Assume that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$, *and* $K(z)$ *is defined by* (2.1). *For a sufficiently large positive integer l, set*

$$\tilde{a} := \{\tilde{a}_n\}_{n\in\mathbb{Z}^0} := \left\{|n|^{\alpha\kappa-1-\frac{2\kappa}{ql}}\right\}_{n\in\mathbb{Z}^0},$$

$$\tilde{f}(x) := \begin{cases} |x|^{\alpha\tau-1+\frac{2\tau}{pl}} & x \in S \\ 0 & x \in \mathbb{R} \setminus S \end{cases},$$

where $S := \{x : |x|^{\operatorname{sgn}\tau} < 1\}$. *Then*

$$(2.14)\quad \tilde{J} := \sum_{n\in\mathbb{Z}^0} \tilde{a}_n \int_{-\infty}^{\infty} K(x^\tau n^\kappa)\tilde{f}(x)\mathrm{d}x = \int_{-\infty}^{\infty} \tilde{f}(x) \sum_{n\in\mathbb{Z}^0} \tilde{a}_n K(x^\tau n^\kappa)\,\mathrm{d}x$$
$$> \frac{I}{|\tau\kappa|}\left(\int_{[-1,1]} K(z)\,|z|^{\alpha-1+\frac{2}{pl}}\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} K(z)\,|z|^{\alpha-1-\frac{2}{ql}}\mathrm{d}z\right).$$

Proof. Write

$$\tilde{J} = \int_{x\in S^-} \tilde{f}(x) \sum_{n\in\mathbb{Z}^+} \tilde{a}_n K(x^\tau n^\kappa)\,\mathrm{d}x + \int_{x\in S^-} \tilde{f}(x) \sum_{n\in\mathbb{Z}^-} \tilde{a}_n K(x^\tau n^\kappa)\,\mathrm{d}x$$
$$+ \int_{x\in S^+} \tilde{f}(x) \sum_{n\in\mathbb{Z}^+} \tilde{a}_n K(x^\tau n^\kappa)\,\mathrm{d}x + \int_{x\in S^+} \tilde{f}(x) \sum_{n\in\mathbb{Z}^-} \tilde{a}_n K(x^\tau n^\kappa)\,\mathrm{d}x$$
$$:= J_1 + J_2 + J_3 + J_4,$$

where $S^+ := \{x : x \in S \cap \mathbb{R}^+\}$, $S^- := \{x : x \in S \cap \mathbb{R}^-\}$.

If $x \in S^-$, $n \in \mathbb{Z}^+$, then we have $x^\tau n^\kappa < 0$. By Lemma 2.1, it can be proved that $G(x^\tau n^\kappa)$ decreases with n ($n \in \mathbb{Z}^+$). Additionally, in views of $\kappa \in (0, 1]\cap\Omega$, it can also be proved that $|n|^{\kappa-1-\frac{2\kappa}{ql}}$ decreases with n ($n \in \mathbb{Z}^+$). It follows therefore that

$$\tilde{a}_n K(x^\tau n^\kappa) = |x|^{\tau(1-\alpha)} G(x^\tau n^\kappa)\,|n|^{\kappa-1-\frac{2\kappa}{ql}}$$

decreases with n ($n \in \mathbb{Z}^+$) for a fixed x ($x \in S^-$), and it implies that

$$J_1 > \int_{x\in S^-} |x|^{\alpha\tau-1+\frac{2\tau}{pl}} \int_1^{\infty} K(x^\tau y^\kappa)\,|y|^{\alpha\kappa-1-\frac{2\kappa}{ql}}\mathrm{d}y\mathrm{d}x := P_1,$$

Similarly, it can be obtained that

$$J_2 > \int_{x\in S^-} |x|^{\alpha\tau-1+\frac{2\tau}{pl}} \int_{-\infty}^{-1} K(x^\tau y^\kappa)\,|y|^{\alpha\kappa-1-\frac{2\kappa}{ql}}\mathrm{d}y\mathrm{d}x := P_2,$$
$$J_3 > \int_{x\in S^+} |x|^{\alpha\tau-1+\frac{2\tau}{pl}} \int_{1}^{\infty} K(x^\tau y^\kappa)\,|y|^{\alpha\kappa-1-\frac{2\kappa}{ql}}\mathrm{d}y\mathrm{d}x := P_3,$$
$$J_4 > \int_{x\in S^+} |x|^{\alpha\tau-1+\frac{2\tau}{pl}} \int_{-\infty}^{-1} K(x^\tau y^\kappa)\,|y|^{\alpha\kappa-1-\frac{2\kappa}{ql}}\mathrm{d}y\mathrm{d}x := P_4.$$

If $\tau < 0$, that is, $\tau \in \Omega\cap\mathbb{R}^-$, then $S^- = S\cap\mathbb{R}^- = (-\infty, -1)$. Let $x^\tau y^\kappa = z$, and observe that $x^{-\frac{\tau}{\kappa}} = -|x|^{-\frac{\tau}{\kappa}}$ $(x < 0)$ and $z^{\frac{1}{\kappa}-1} = |z|^{\frac{1}{\kappa}-1}$ $(z < 0)$, then we have

$$(2.15)\qquad P_1 = \int_{-\infty}^{-1} |x|^{\alpha\tau-1+\frac{2\tau}{pl}} \int_1^{\infty} K(x^\tau y^\kappa)\,|y|^{\alpha\kappa-1-\frac{2\kappa}{ql}}\mathrm{d}y\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{\kappa}\int_{-\infty}^{-1} |x|^{-1+\frac{2\tau}{l}} \int_{-\infty}^{x^\tau} K(z)\,|z|^{\alpha-1-\frac{2}{ql}}\,dzdx \\
&= \frac{1}{\kappa}\int_{-\infty}^{-1} |x|^{-1+\frac{2\tau}{l}} \int_{-\infty}^{-1} K(z)\,|z|^{\alpha-1-\frac{2}{ql}}\,dzdx \\
&+ \frac{1}{\kappa}\int_{-\infty}^{-1} |x|^{-1+\frac{2\tau}{l}} \int_{-1}^{x^\tau} K(z)\,|z|^{\alpha-1-\frac{2}{ql}}\,dzdx \\
&= \frac{l}{2\,|\tau\kappa|}\int_{-\infty}^{-1} K(z)\,|z|^{\alpha-1-\frac{2}{ql}}\,dz \\
&+ \frac{1}{\kappa}\int_{-\infty}^{-1} |x|^{-1+\frac{2\tau}{l}} \int_{-1}^{x^\tau} K(z)\,|z|^{\alpha-1-\frac{2}{ql}}\,dzdx.
\end{aligned}
$$

It follows from Fubini's theorem that

$$
\begin{aligned}
(2.16)\qquad &\int_{-\infty}^{-1} |x|^{-1+\frac{2\tau}{l}} \int_{-1}^{x^\tau} K(z)\,|z|^{\alpha-1-\frac{2}{ql}}\,dzdx \\
&= \int_{-1}^{0} K(z)\,|z|^{\alpha-1-\frac{2}{ql}} \int_{-\infty}^{z^{1/\tau}} |x|^{-1+\frac{2\tau}{l}}\,dxdz \\
&= \frac{l}{2\,|\tau|}\int_{-1}^{0} K(z)\,|z|^{\alpha-1+\frac{2}{pl}}\,dz.
\end{aligned}
$$

Plugging (2.16) back into (2.15), we obtain

$$
P_1 = \frac{l}{2\,|\tau\kappa|}\left(\int_{-\infty}^{-1} K(z)\,|z|^{\alpha-1-\frac{2}{ql}}\,dz + \int_{-1}^{0} K(z)\,|z|^{\alpha-1+\frac{2}{pl}}\,dz\right).
$$

Similarly, it can be obtained that $P_4 = P_1$, and

$$
P_2 = P_3 = \frac{l}{2\,|\tau\kappa|}\left(\int_{1}^{\infty} K(z)\,|z|^{\alpha-1-\frac{2}{ql}}\,dz + \int_{0}^{1} K(z)\,|z|^{\alpha-1+\frac{2}{pl}}\,dz\right).
$$

It implies that

$$
\begin{aligned}
\tilde{J} &> P_1 + P_2 + P_3 + P_4 \\
&= \frac{l}{|\tau\kappa|}\left(\int_{[-1,1]} K(z)\,|z|^{\alpha-1+\frac{2}{pl}}\,dz + \int_{\mathbb{R}\setminus[-1,1]} K(z)\,|z|^{\alpha-1-\frac{2}{ql}}\,dz\right).
\end{aligned}
$$

Hence, Lemma 2.3 is proved when $\tau < 0$. If $\tau > 0$. it can also be proved that (2.14) holds true. The proof of Lemma 2.3 is completed. □

3 Main Results

Theorem 3.1. *Let* $\delta \in \{1, -1\}$*, and*

$$
\Omega := \left\{ t : t = \frac{2i+1}{2j+1}, i, j \in \mathbb{Z} \right\}.
$$

Suppose that $\alpha \in (0,1)$, $\tau \in \Omega$, $\kappa \in (0,1] \cap \Omega$, $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, *and* α, β, γ *satisfy* $\beta < \gamma$ *and* $\alpha + \beta < \min\{1, \gamma\}$. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Assume that* $\mu(x) = |x|^{p(1-\alpha\tau)-1}$, $\nu_n = |n|^{q(1-\alpha\kappa)-1}$, *where* $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$. *Let* $f(x), a_n \geq 0$ *be such that* $f(x) \in L_{p,\mu}(\mathbb{R})$, *and* $a = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. *Let* $\Phi(z) = \cot z$, *and* $K(z)$ *be defined by* (2.1). *Then the following inequalities hold and are equivalent:*

$$(3.1)\quad J := \sum_{n\in\mathbb{Z}^0} a_n \int_{-\infty}^{\infty} K(x^\tau n^\kappa) f(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} K(x^\tau n^\kappa) a_n \mathrm{d}x$$
$$< \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right]\|f\|_{p,\mu}\|a\|_{q,\nu},$$

$$(3.2)\quad L_1 := \sum_{n\in\mathbb{Z}^0} |n|^{p\alpha\kappa-1}\left(\int_{-\infty}^{\infty} K(x^\tau n^\kappa) f(x)\mathrm{d}x\right)^p$$
$$< \left\{\frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right]\right\}^p \|f\|^p_{p,\mu},$$

$$(3.3)\quad L_2 := \int_{-\infty}^{\infty} |x|^{q\alpha\tau-1}\left(\sum_{n\in\mathbb{Z}^0} K(x^\tau n^\kappa) a_n\right)^q \mathrm{d}x$$
$$< \left\{\frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right]\right\}^q \|a\|^q_{q,\nu},$$

where the constant $\frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right]$ *in* (3.1), (3.2) *and* (3.3) *is the best possible.*

Proof. Let $\tilde{K}(x^\tau y^\kappa) := K(x^\tau n^\kappa)$, $g(y) := a_n$, and $h(y) := n$ for $y \in [n-1, n)$, $n \in \mathbb{Z}^+$. Let $\tilde{K}(x^\tau y^\kappa) := K(x^\tau n^\kappa)$, $g(y) := a_n$, and $h(y) := |n|$ for $y \in [n, n+1)$, $n \in \mathbb{Z}^-$. By Hölder's inequality, we have

$$(3.4)\quad \sum_{n\in\mathbb{Z}^0} a_n \int_{-\infty}^{\infty} K(x^\tau n^\kappa) f(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} K(x^\tau n^\kappa) a_n \mathrm{d}x$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{K}(x^\tau y^\kappa) f(x) g(y)\mathrm{d}x\mathrm{d}y$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\tilde{K}(x^\tau y^\kappa)\right]^{1/p} [h(y)]^{(\alpha\kappa-1)/p}|x|^{(1-\alpha\tau)/q} f(x)$$
$$\times \left[\tilde{K}(x^\tau y^\kappa)\right]^{1/q} |x|^{(\alpha\tau-1)/q}[h(y)]^{(1-\alpha\kappa)/p} g(y)\mathrm{d}x\mathrm{d}y$$
$$\leq \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{K}(x^\tau y^\kappa)[h(y)]^{\alpha\kappa-1}|x|^{p(1-\alpha\tau)/q} f^p(x)\mathrm{d}y\mathrm{d}x\right)^{1/p}$$
$$\times \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{K}(x^\tau y^\kappa)|x|^{\alpha\tau-1}[h(y)]^{q(1-\alpha\kappa)/p} g^q(y)\mathrm{d}x\mathrm{d}y\right)^{1/q}$$

$$= \left[\int_{-\infty}^{\infty} \psi(x)\, |x|^{p(1-\alpha\tau)/q} f^p(x)\mathrm{d}x\right]^{1/p} \left[\sum_{n\in\mathbb{Z}^0} \omega(n)\, |n|^{q(1-\alpha\kappa)/p}\, a_n^q\right]^{1/q},$$

where

(3.5) $$\psi(x) = \sum_{n\in\mathbb{Z}^0} K\left(x^\tau n^\kappa\right) |n|^{\alpha\kappa-1},$$

(3.6) $$\omega(n) = \int_{-\infty}^{\infty} K\left(x^\tau n^\kappa\right) |x|^{\alpha\tau-1}\,\mathrm{d}x.$$

Since $\kappa \leq 1$, it can be shown that $|n|^{\kappa-1}$ decreases monotonically if $n \in \mathbb{Z}^+$, and increases monotonically if $n \in \mathbb{Z}^-$. Moreover, by Lemma 2.1, and observing that $\tau \in \Omega$ and $\kappa \in (0, 1] \cap \Omega$, it can be proved that whether $x \in \mathbb{R}^+$ or $x \in \mathbb{R}^-$, $G(x^\tau n^\kappa)$ decreases monotonically with n if $n \in \mathbb{Z}^+$, and increases monotonically with n if $n \in \mathbb{Z}^-$. Therefore, for a fixed x,

$$K\left(x^\tau n^\kappa\right) |n|^{\alpha\kappa-1} = |x|^{\tau-\alpha\tau}\, G\left(x^\tau n^\kappa\right) |n|^{\kappa-1}$$

decreases monotonically with n if $n \in \mathbb{Z}^+$, and increases monotonically with n if $n \in \mathbb{Z}^-$. It follows therefore that

$$\psi(x) = \sum_{n\in\mathbb{Z}^0} K\left(x^\tau n^\kappa\right) |n|^{\alpha\kappa-1} < \int_{-\infty}^{\infty} K\left(x^\tau y^\kappa\right) |y|^{\alpha\kappa-1}\,\mathrm{d}y,$$

Supposing that $x < 0$, and Observing that $\tau \in \Omega$ and $\kappa \in (0, 1] \cap \Omega$, we get $x^{-\frac{\tau}{\kappa}} = -|x|^{-\frac{\tau}{\kappa}}$ and $z^{\frac{1}{\kappa}-1} = |z|^{\frac{1}{\kappa}-1}$. Letting $x^\tau y^\kappa = z$, it follows that

(3.7) $$\psi(x) < \int_{-\infty}^{\infty} K\left(x^\tau y^\kappa\right) |y|^{\alpha\kappa-1}\,\mathrm{d}y = \frac{|x|^{-\alpha\tau}}{\kappa} \int_{-\infty}^{\infty} K(z)\, |z|^{\alpha-1}\,\mathrm{d}z.$$

By similar discussion, it can also be proved that (3.7) is valid for $x > 0$. Plugging (2.8) into (3.7), we get

(3.8) $$\psi(x) < \frac{\pi\, |x|^{-\alpha\tau}}{\kappa\gamma}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right].$$

Additionally, it can also be obtained that

(3.9) $$\omega(n) = \frac{\pi\, |n|^{-\alpha\kappa}}{|\tau|\,\gamma}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right].$$

Plugging (3.8) and (3.9) back into (3.4), we obtain (3.1). In what follows, it is to be proved that (3.2) and (3.3) hold under the condition that inequality (3.1) holds. In fact, Let $b = \{b_n\}_{n\in\mathbb{N}^0}$, where

$$b_n := |n|^{p\alpha\kappa-1}\left(\int_{-\infty}^{\infty} K\left(x^\tau n^\kappa\right) f(x)\,\mathrm{d}x\right)^{p-1},$$

then

$$(3.10)\qquad L_1 = \sum_{n\in Z^0} |n|^{p\alpha\kappa-1}\left[\int_{-\infty}^{\infty} K(x^\tau n^\kappa) f(x)\,dx\right]^p$$
$$= \sum_{n\in Z^0} b_n \int_{-\infty}^{\infty} K(x^\tau n^\kappa) f(x)\,dx$$
$$< \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right]\|f\|_{p,\mu}\|b\|_{q,v}$$
$$= \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right]\|f\|_{p,\mu}L_1^{1/q}.$$

It follows from (3.10) that (3.2) holds true. Similarly, inequality (3.3) can be proved. In fact, setting

$$g(x) := |x|^{q\alpha\tau-1}\left[\sum_{n\in Z^0} K(x^\tau n^\kappa)\, a_n\right]^{q-1},$$

and using (3.1), it follows that

$$(3.11)\qquad L_2 = \int_{-\infty}^{\infty} |x|^{q\alpha\tau-1}\left[\sum_{n\in Z^0} K(x^\tau n^\kappa)\, a_n\right]^q dx$$
$$= \int_{-\infty}^{\infty} g(x)\sum_{n\in Z^0} K(x^\tau n^\kappa)\, a_n\,dx$$
$$< \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right]\|g\|_{p,\mu}\|a\|_{q,v}$$
$$= \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right]\|a\|_{q,v}L_2^{1/p}.$$

Therefore, (3.3) follows obviously. Furthermore, it can be proved that (3.1) holds true when inequality (3.2) or (3.3) is valid. In fact, assuming (3.2) holds true, it follows from Hölder's inequality that

$$(3.12)\qquad J = \sum_{n\in Z^0} \left[|n|^{\alpha\tau-1/p}\int_{-\infty}^{\infty} K(x^\tau n^\kappa) f(x)\,dx\right]\left[a_n|n|^{-\alpha\tau+1/p}\right]$$
$$\le L_1^{1/p}\left[\sum_{n\in Z^0} a_n^q |n|^{q(1-\alpha\tau)-1}\right]^{1/q} = L_1^{1/p}\|a\|_{q,v}.$$

Apply inequality (3.2) to (3.12), then we arrive at (3.1). Similarly, if we suppose that inequality (3.3) holds true, it can also be proved that (3.1) is valid. Thus, inequalities (3.1), (3.2) and (3.3) are equivalent.

In what follows, it will be proved that the constant factors in (3.1), (3.2) and (3.3) are the best possible. In fact, suppose that there exists a constant C which satisfies

$$0 < C \le \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right], \tag{3.13}$$

so that

$$J = \sum_{n\in \mathbb{Z}^0} a_n \int_{-\infty}^{\infty} K(x^{\tau} n^{\kappa}) f(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x) \sum_{n\in \mathbb{Z}^0} K(x^{\tau} n^{\kappa})\, a_n \mathrm{d}x \tag{3.14}$$
$$< C\|f\|_{p,\mu}\|a\|_{q,\nu}.$$

Replace a_n and $f(x)$ in (3.14) with $\tilde{a}_n$ and $\tilde{f}(x)$ defined in Lemma 2.3, repectively, and use (2.14), then we have

$$\int_{[-1,1]} K(z)\,|z|^{\alpha-1+\frac{2}{pl}}\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} K(z)\,|z|^{\alpha-1-\frac{2}{ql}}\mathrm{d}z \tag{3.15}$$
$$< \frac{|\tau\kappa|}{l}\tilde{J} < \frac{|\tau\kappa|\,C}{l}\|\tilde{f}\|_{p,\mu}\|\tilde{a}\|_{q,\nu}$$
$$= \frac{|\tau\kappa|\,C}{l}\left(2\int_{S^+} x^{\frac{2\tau}{l}-1}\mathrm{d}x\right)^{\frac{1}{p}}\left(2+2\sum_{n=2}^{\infty} n^{-\frac{2\kappa}{l}-1}\right)^{\frac{1}{q}}$$
$$< \frac{2\,|\tau\kappa|\,C}{l}\left(\int_{S^+} x^{\frac{2\tau}{l}-1}\mathrm{d}x\right)^{\frac{1}{p}}\left(1+\int_1^{\infty} x^{-\frac{2\kappa}{l}-1}\mathrm{d}x\right)^{\frac{1}{q}}$$
$$= 2\,|\tau\kappa|\,C\left(\frac{1}{2\,|\tau|}\right)^{\frac{1}{p}}\left(\frac{1}{l}+\frac{1}{2\kappa}\right)^{\frac{1}{q}}.$$

Apply Fatou's lemma to (3.15), and use (2.8), then it follows that

$$\frac{\pi}{\gamma}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right] = \int_{-\infty}^{\infty} K(z)\,|z|^{\alpha-1}\,\mathrm{d}z$$
$$= \int_{[-1,1]} \lim_{l\to\infty} K(z)\,|z|^{\alpha-1+\frac{2}{pl}}\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} \lim_{l\to\infty} K(z)\,|z|^{\alpha-1-\frac{2}{ql}}\mathrm{d}z$$
$$\leqslant \lim_{l\to\infty}\left[\int_{[-1,1]} K(z)\,|z|^{\alpha-1+\frac{2}{pl}}\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} K(z)\,|z|^{\alpha-1-\frac{2}{ql}}\mathrm{d}z\right]$$
$$\leqslant \lim_{l\to\infty} 2\,|\tau\kappa|\,C\left(\frac{1}{2\,|\tau|}\right)^{\frac{1}{p}}\left(\frac{1}{l}+\frac{1}{2\kappa}\right)^{\frac{1}{q}} = C\,|\tau|^{\frac{1}{q}}\kappa^{\frac{1}{p}}.$$

It follows that

$$C \ge \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right]. \tag{3.16}$$

Combine (3.13) and (3.16), then we have

$$C = \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right].$$

Hence, it is proved that the constant factor in inequality (3.1) is the best possible. Observing that inequalities (3.1), (3.2) and (3.3) are equivalent, it can also be proved that the constant factors in (3.2) and (3.3) are the best possible. Theorem 3.1 is proved. □

4 Corollaries

Let $\gamma = 3\beta$, $\tau = \kappa = 1$ in Theorem 3.1. Then (3.1) is transformed into the following Hilbert-type inequality with a non-homogeneous kernel.

Corollary 4.1. *Let $\delta \in \{1, -1\}$, and*

$$\Omega := \left\{t : t = \frac{2i+1}{2j+1}, i, j \in \mathbb{Z}\right\}$$

Suppose that $\alpha \in (0,1)$, $\beta \in \Omega$, and α, β satisfy $0 < \alpha < 2\beta$ and $\alpha + \beta < 1$. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $\mu(x) = |x|^{p(1-\alpha)-1}$, $\nu_n = |n|^{q(1-\alpha)-1}$, where $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$. Let $f(x), a_n \geq 0$ be such that $f(x) \in L_{p,\mu}(\mathbb{R})$, and $a = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. Let $\Phi(z) = \cot z$. Then

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{1 + \delta\,(xn)^{\beta} + (xn)^{2\beta}}\,dx < \frac{\pi}{3\beta}\left[\Phi\left(\frac{\alpha\pi}{6\beta}\right) - \Phi\left(\frac{(\alpha+4\beta)\pi}{6\beta}\right)\right]\|f\|_{p,\mu}\|a\|_{q,\nu}, \tag{4.1}$$

where the constant factor $\frac{\pi}{3\beta}\left[\Phi\left(\frac{\alpha\pi}{6\beta}\right) - \Phi\left(\frac{(\alpha+4\beta)\pi}{6\beta}\right)\right]$ in (4.1) is the best possible.

Set $\alpha = \frac{\beta}{2}$ in Corollary 4.1, then $\beta \in \Omega$, and $0 < \beta < \frac{2}{3}$. Since $\Phi\left(\frac{\pi}{12}\right) = 2 + \sqrt{3}$, we obtain

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{1 + \delta\,(xn)^{\beta} + (xn)^{2\beta}}\,dx < \frac{(4+\sqrt{3})\pi}{3\beta}\|f\|_{p,\mu}\|a\|_{q,\nu}, \tag{4.2}$$

where $\mu(x) = |x|^{p(1-\beta/2)-1}$, $\nu_n = |n|^{q(1-\beta/2)-1}$. Letting $\delta = 1$, we have (1.7).

Set $\alpha = \beta$ in Corollary 4.1, then $\beta \in \Omega$, $0 < \beta < \frac{1}{2}$ and (4.1) reduces to the following inequality.

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{1 + \delta\,(xn)^{\beta} + (xn)^{2\beta}}\,dx < \frac{2\sqrt{3}\pi}{3\beta}\|f\|_{p,\mu}\|a\|_{q,\nu}, \tag{4.3}$$

where $\mu(x) = |x|^{p(1-\beta)-1}$, $\nu_n = |n|^{q(1-\beta)-1}$.

Set $\alpha = \frac{3\beta}{2}$ in Corollary 4.1, then $\beta \in \Omega$, and $0 < \beta < \frac{2}{5}$. In view of $\Phi\left(\frac{11\pi}{12}\right) = 2 + \sqrt{3}$, we arrive at

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in \mathbb{Z}^0} \frac{a_n}{1+\delta\,(xn)^{\beta} + (xn)^{2\beta}}\,dx < \frac{(4+\sqrt{3})\pi}{3\beta}\|f\|_{p,\mu}\|a\|_{q,\nu}, \tag{4.4}$$

where $\mu(x) = |x|^{p(1-3\beta/2)-1}$, $\nu_n = |n|^{q(1-3\beta/2)-1}$.

Let $\alpha = \frac{\gamma-\beta}{2}$, $\tau = \kappa = 1$ in Theorem 3.1. Then it can be obtained another Hilbert-type inequality with a non-homogeneous kernel.

Corollary 4.2. *Let $\delta \in \{1, -1\}$, and*

$$\Omega := \left\{ t : t = \frac{2i+1}{2j+1}, i, j \in \mathbb{Z} \right\}.$$

Suppose that $\beta, \gamma \in \Omega$, $0 < \beta < \gamma$ and $\beta + \gamma < 2$. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $\mu(x) = |x|^{p(\beta-\gamma+2)/2-1}$, $\nu_n = |n|^{q(\beta-\gamma+2)/2-1}$, where $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$. Let $f(x), a_n \geq 0$ be such that $f(x) \in L_{p,\mu}(\mathbb{R})$, and $a = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. Let $\Phi(z) = \cot z$. Then

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{1-\delta\,(xn)^{\beta}}{1-\delta\,(xn)^{\gamma}} a_n\,dx < \frac{2\pi}{\gamma}\Phi\left(\frac{(\gamma-\beta)\pi}{4\gamma}\right)\|f\|_{p,\mu}\|a\|_{q,\nu}, \tag{4.5}$$

where the constant factor $\frac{2\pi}{3\beta}\Phi\left(\frac{(\gamma-\beta)\pi}{4\gamma}\right)$ in (4.5) is the best possible.

Letting $\gamma = (2k+1)\beta$, $k \in \mathbb{N}^+$, we have $0 < (k+1)\beta < 1$, $\beta \in \Omega$, and (4.5) is transformed into the following inequality.

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\sum_{j=0}^{2k} \delta^{2k-j}(xn)^{j\beta}}\,dx < \frac{2\pi}{(2k+1)\beta}\Phi\left(\frac{k\pi}{4k+2}\right)\|f\|_{p,\mu}\|a\|_{q,\nu}, \tag{4.6}$$

where $\mu(x) = |x|^{p(1-k\beta)-1}$, $\nu_n = |n|^{q(1-k\beta)-1}$.

Setting $k = 1$ in (4.6), we can also get (4.3). Moreover, Setting $k = 2$ in (4.6), we have $0 < \beta < \frac{1}{3}$, $\beta \in \Omega$. It follows that

$$\int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}^0} \frac{a_n f(x)}{1+\delta(xn)^{\beta}+(xn)^{2\beta}+\delta(xn)^{3\beta}+(xn)^{4\beta}}\,dx < \frac{2\pi}{5\beta}\Phi\left(\frac{\pi}{5}\right)\|f\|_{p,\mu}\|a\|_{q,\nu}, \tag{4.7}$$

where $\mu(x) = |x|^{p(1-2\beta)-1}$, $\nu_n = |n|^{q(1-2\beta)-1}$.

Let $\gamma = 3\beta$, $\tau = -1$, $\kappa = 1$ in Theorem 3.1, and replace $f(x)x^{2\beta}$ with $f(x)$. Then it can be obtained the following Hilbert-type inequality with a homogeneous kernel of degree 2β.

Corollary 4.3. *Let* $\delta \in \{1, -1\}$, *and*

$$\Omega := \left\{ t : t = \frac{2i+1}{2j+1}, i, j \in \mathbb{Z} \right\}.$$

Suppose that $\alpha \in (0, 1)$, $\beta \in \Omega$, *and* α, β *satisfy* $0 < \alpha < 2\beta$ *and* $\alpha + \beta < 1$. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Assume that* $\mu(x) = |x|^{p(1+\alpha-2\beta)-1}$, $v_n = |n|^{q(1-\alpha)-1}$, *where* $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$. *Let* $f(x), a_n \geq 0$ *be such that* $f(x) \in L_{p,\mu}(\mathbb{R})$, *and* $a = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,v}$. *Let* $\Phi(z) = \cot z$. *Then*

$$(4.8) \qquad \int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{x^{2\beta} + \delta(xn)^{\beta} + n^{2\beta}} dx < \frac{\pi}{3\beta}\left[\Phi\left(\frac{\alpha\pi}{6\beta}\right) - \Phi\left(\frac{(\alpha+4\beta)\pi}{6\beta}\right)\right] \|f\|_{p,\mu} \|a\|_{q,v},$$

where the constant factor $\frac{\pi}{3\beta}\left[\Phi\left(\frac{\alpha\pi}{6\beta}\right) - \Phi\left(\frac{(\alpha+4\beta)\pi}{6\beta}\right)\right]$ *in (4.8) is the best possible.*

Set $\alpha = \frac{\beta}{2}$ in Corollary 4.3, then $\beta \in \Omega$, and $0 < \beta < \frac{2}{3}$. Since $\Phi\left(\frac{\pi}{12}\right) = 3 + \sqrt{3}$, we obtain the following inequality.

$$(4.9) \qquad \int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{x^{2\beta} + \delta(xn)^{\beta} + n^{2\beta}} dx < \frac{(4+\sqrt{3})\pi}{3\beta} \|f\|_{p,\mu} \|a\|_{q,v},$$

where $\mu(x) = |x|^{p(1-3\beta/2)-1}$, $v_n = |n|^{q(1-\beta/2)-1}$. Letting $\delta = -1$, we obtain inequality (1.8).

Set $\alpha = \beta$ in Corollary 4.3, then $\beta \in \Omega$, and $0 < \beta < \frac{1}{2}$ and (4.8) is transformed into the following inequality.

$$(4.10) \qquad \int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{x^{2\beta} + \delta(xn)^{\beta} + n^{2\beta}} dx < \frac{2\sqrt{3}\pi}{3\beta} \|f\|_{p,\mu} \|a\|_{q,v},$$

where $\mu(x) = |x|^{p(1-\beta)-1}$, $v_n = |n|^{q(1-\beta)-1}$

Let $\alpha = \frac{\gamma-\beta}{2}$, $\tau = -1$, $\kappa = 1$ in Theorem 3.1, and replace $f(x)x^{\gamma-\beta}$ with $f(x)$. Then it can be obtained the following Hilbert-type inequality involving a homogeneous kernel with degree $\gamma - \beta$.

Corollary 4.4. *Let* $\delta \in \{1, -1\}$, *and*

$$\Omega := \left\{ t : t = \frac{2i+1}{2j+1}, i, j \in \mathbb{Z} \right\}.$$

Suppose that $\beta, \gamma \in \Omega$, $0 < \beta < \gamma$ *and* $\beta + \gamma < 2$. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Assume that* $\mu(x) = |x|^{p(\beta-\gamma+2)/2-1}$, $v_n = |n|^{q(\beta-\gamma+2)/2-1}$, *where* $n \in \mathbb{Z}^0 :=$

$\mathbb{Z}\setminus\{0\}$. *Let* $f(x), a_n \geq 0$ *be such that* $f(x) \in L_{p,\mu}(\mathbb{R})$, *and* $a = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. *Let* $\Phi(z) = \cot z$. *Then*

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{x^\beta - \delta n^\beta}{x^\gamma - \delta n^\gamma} a_n dx < \frac{2\pi}{\gamma}\Phi\left(\frac{(\gamma-\beta)\pi}{4\gamma}\right)\|f\|_{p,\mu}\|a\|_{q,\nu}, \tag{4.11}$$

where the constant factor $\frac{2\pi}{\gamma}\Phi\left(\frac{(\gamma-\beta)\pi}{4\gamma}\right)$ *in* (4.11) *is the best possible.*

Letting $\gamma = (2k+1)\beta$, $k \in \mathbb{N}^+$, we have $0 < (k+1)\beta < 1$, $\beta \in \Omega$, and (4.11) is transformed into the following inequality.

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\sum_{j=0}^{2k} \delta^{2k-j} x^{j\beta} n^{(2k-j)\beta}} dx < \frac{2\pi}{(2k+1)\beta}\Phi\left(\frac{k\pi}{4k+2}\right)\|f\|_{p,\mu}\|a\|_{q,\nu}, \tag{4.12}$$

where $\mu(x) = |x|^{p(1-k\beta)-1}$, $\nu_n = |n|^{q(1-k\beta)-1}$.

Setting $k = 1$, and $\delta = 1$ in (4.12), it can also be obtained (4.10). Additionally, let $k = 2$ in (4.12), then $0 < \beta < \frac{1}{8}$ $\beta \in \Omega$, and it follows that

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{x^{4\beta} + x^{3\beta}n^\beta + (xn)^{2\beta} + x^\beta n^{3\beta} + n^{4\beta}} dx < \frac{2\pi}{5}\Phi\left(\frac{\pi}{5}\right)\|f\|_{p,\mu}\|a\|_{q,\nu}. \tag{4.13}$$

where $\mu(x) = |x|^{p(1-2\beta)-1}$, $\nu_n = |n|^{q(1-2\beta)-1}$.

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