CONCEPS OF RELATION/FUNCTION ALGEBRAIC STRUCTURE

Mukesh Kumar Pal[1] & Shambhu Kumar[2]

1. Department of Mathematics, Veer Kunwar singh University, Ara-802301, (Bihar).

2. Department of Mathematics, Veer Kunwar singh University, Ara-802301, (Bihar).

Abstract

In this paper, we have to discuss about the concept of relation/function of algebraic structure and its extension to fuzzy set, is a generalization of crisp sets. Relations between elements of crisp set can be extended to fuzzy Relation and Relation will be considered as fuzzy sets.

Keyword: crisp set, algebraic structure, cartesian product, linguistic terms.

1. Introduction:

Fuzzy sets are generalizations of conventional set theory introduced by Zadeh (1965) as a mathematical way to represent vagueness in everyday life. A fuzzy set assigns to each possible individual in the universe of discourse, a value representing its grade of membership in the set. It is concerned with the degree to which events occur rather than the likelihood of their occurrence. Fuzzy logic is most successful in situations with very complex models, where understanding is strictly limited and where human reasoning, human perception, human decision making are inextricably involved. Fuzzy sets play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, decision making and abstraction. Applications of fuzzy sets in various fields are discussed in Timothy (1997) and George J. Klir and Bo Yuan (1997).

In conventional set theory, elements of a set satisfy precise properties. In crisp sets an element x in the universe X is either a member or not a member of some crisp set A. This binary issue of membership can be represented mathematically with a function called characteristic. Where Crisp sets handle black and white concepts. However, everyday life abounds in innumerable vague concepts like 'young', 'old', 'hot', 'intelligent' and linguistic terms like 'few', 'very few', 'almost all', etc.,. The major limitation of classical set theory concept is that it fails to define such vague concepts which are favorably addressed by the fuzzy set theory. A unique advantage of the fuzzy set 1 theory is that its ability to generalize 0 and 1 membership values of a crisp set to a membership function of a fuzzy set.

1.1. Crisp Relation:

Definition (Product set): The product set AxB of two non empty sets A and B is a set whose elements are pair elements where 1^{st} element comes from the 1^{st} set A and the 2^{nd} elements from the 2^{nd} sets B.

The concept of cartesian product can be standered to n set for an arbitrary number of sets A_1, A_2 …….. An the set of all n-tuples (a_1….,an) such that, is called the cartesion product written as A_1XA_2 X …… An or$a_1 \in A_1$ $a_2 \in A_2$ …….an

$$\sum_{i=l}^{n} Ai$$

When all the An sets and identical and equal to A, the cantesian product A_1XA_2 x A_3 x A_4 … x A_n is denoted by A^n the product is used for composition of sets or relations.

Example of a cartesion product set A x B, When

A = {a_1, a_2, a_3}, B = {b_1,b_2) cantesion produced

A x B – { (a_1 b_2) (a_1 b_2)(a_2 b_1)(a_2 b_2)(a_3 b_1)(a_3 b_2)}

as shown in fig 1

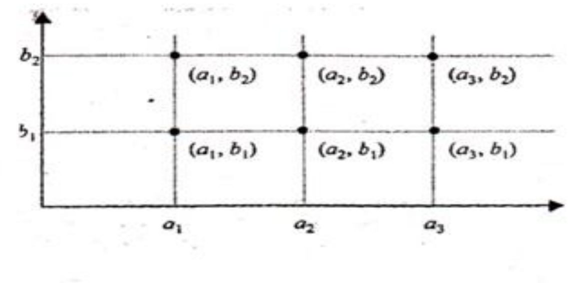


Fig. 1 Product set A x B

The example of a cartesion product A x A wher A = {a_1,a_2,a_3}

A x A = { (a_1 a_1) (a_1 a_2)(a_1 a_3)(a_2 a_1)(a_2 a_2)(a_2 a_3) (a_3 a_1) (a_3 a_2)(a_3 a_3)

As shown in fig 2

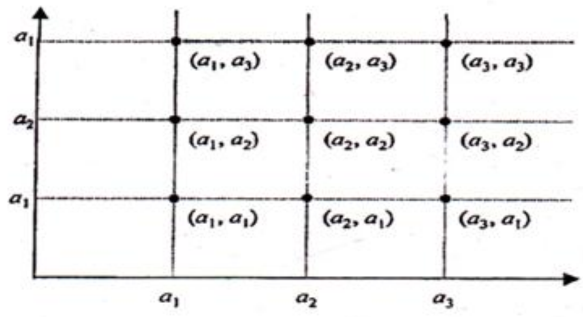


Fig. 2. Cartesian product $A \times A$

1.2. Definition of Relation

Definition (Binary Relation) if A and B are two sets and there is a specific property between elements x of A and y of B, this property can be described using the ordered pair (x, y). A set of such (x, y) pairs, $x \in A$ and $y \epsilon B$ is called a relation R.

R= { $(x, y)| x\epsilon\, a, y\, \epsilon B$}

R is a binary relation and s subset of A x B .

The term "x is in relation R with y" is denoted as $(x, y) \in$ Ror xRy with R ⊆AxB.

If $(x, y) \notin$ R, x is not in relation R with y.

If A = B or R is a relation form A to A, it is written

(x, x) ∈R or $xRx$for R⊆ A x A

Definition (n-any relation) for sets $A_1, A_2, A_3 \ldots\ldots A_n$. the relation among elements $x_1, \in$A1, $x_2, \in$A2,$x_3, \in$A3.......$x_n, \in$A1. An can be described by n-Tuple.

$(x_1, x_2, \ldots \ldots \ldots . x_n)$ A collection of such n –tuple

$(x_1, x_2, x_3 \ldots \ldots . . x_n)$in a relation R among.

$A_1, A_2, A_3 \ldots\ldots A_n$. That is

$(x_1, x_2, x_3 \ldots \ldots . . x_n) \in R$

$R \subseteq A_1 \times A_2 \times A_3 \times \ldots\ldots \times A_n$

Definition (Domain and range) Let R stand. For a relation between A and B. The domain and range of this relation are defined as follows (Fig 3)

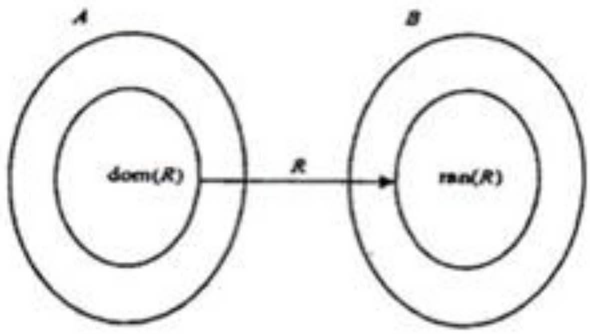


Fig. 3. Domain and range

Dom (R) = { $x|x$ ∈A;{ x, y) ∈ R For some $y \in B$}

Ran (R) = { $y|y$ ∈B;{ x, y) ∈ R For some $x \in A$ }

Here we call set A as support of dom (R) and B as support of ran (R) dom (R) = A results in completely specified and dom (R)⊆ A incompletely specified.

The relation R⊆A x Bis set of ordered pairs (x, y). Thus if we have a certain element x in A, we can find y to B i.e.

The mapped image of A. we say "y is the mapping of x" (Fig 4)

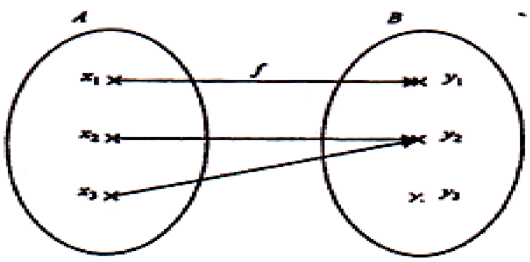


Fig. 4. Mapping $y = f(x)$

If we express this mapping as f, y is called the image of x which is denoted as $f(x)$

$$R = \{ (x, y) \mid x \in A, y \in B, y = f(x)\} \quad \text{or } f : A \to B$$

So we might say ran(R) is the set gathering of these f(x)

$$\text{Ran (R)} = f(A) = \{f(x) | x \in A\}$$

1.3. Properties of Relation:

(a) One –to-many

R is said to be one – to – many if $\exists x \in A, y_1, y_2 \in B (x, y) \in R\ (x, y_2) \in R$,Which is a relation But not a function as soon in picturial diagram as shown in fig 5

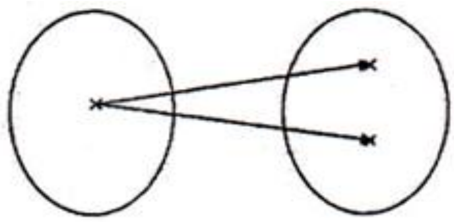


Fig. 5. One-to-many relation (*not a function*)

(b) Surjection (many-to-one)

R is said to be a surjection if f(A) = B or range (R) = B

$$\forall y \in B, \exists\ x \in A, y = f(x)$$

even if $X_1 \neq X_2$, $f(x_1) = f(x_2)$ can hold this many relation may be up two tipes First many on on to relation and many one into relation as shown in Fig 6 and Fig 7 respectively.

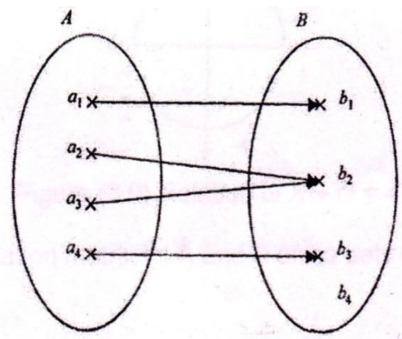


Fig (6)Binary many one- on to relation from A To B

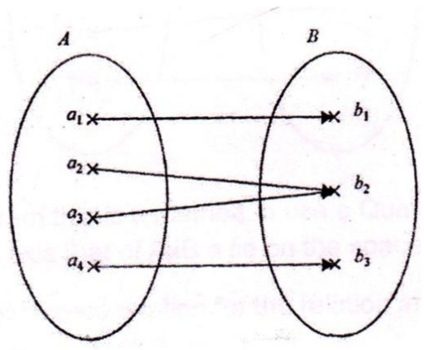


Fig (7)Binary many one- in to relation from A To B

A Relation areform A to B is called bijection or one to one correspondence. If it is both a surjection an injection. That is if the number of elements in A and B are equal and the relation is on to That is one-one on To relation is called bijectionrelation or one-one correspondence.

1.4. Methodof Representation of Relation

There are four methods of representing Relation between sets A and B

1. Bu partigraph this is elestrated A and B in figure and Representing the relation by drawing arcs or agaes as figures 6

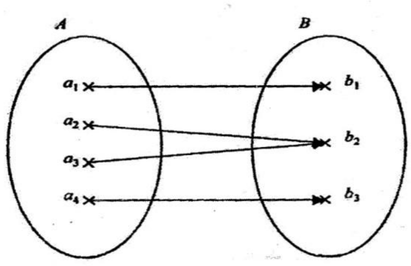


Fig (8) Partigraph

2. By cordinate diagram This is a method to use a quardinate diagram by plotting members of A on X axis that of B and y axisand then the members of A X B a lie on the space.

   Fig (9) shows this type of Representation for the relation are namely $x^2 + y^2$ = 4 where x ∈A and$y$ ∈B as shown

   In figure (9) below

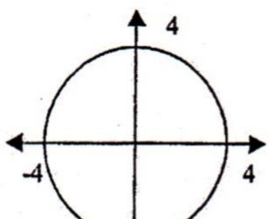


Fig (9) Relation of $x^2 + y^2 = 4$

3. By manipulating relation matrix

Let A and B finite sets having M and N element respectively.

Assuming are is a relation between A and B we many represent the relation by matrix

M_R = (mij) which is defined as follows

M_R = (mij)

$$Mij = \begin{cases} l \left(a_i, b_j\right) \in R \\ o \left(a_i, b_j\right) \in R \end{cases}$$

$$i = 1,2,3, \dots m$$

$$j = 1,2,3, \dots\dots n$$

Such matrix is called a relation matrix and that of the relation in (fig 10) is given in the

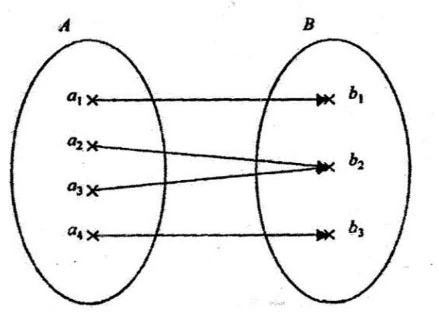


Fig (10)Matrix form as follows.

R	b_1	b_2	b_3
a_1	1	0	0
a_2	0	1	0
a_3	0	1	0
a_4	0	0	1

4. By Directed graph or digraph method elements are represented as nodes and relations between elements as directed edges.
   A = { 1,2,3,4} and R = {(1,1),(1,2),(2,1),(2,2),(1,3), (2,4),(4,1) } for instance. Fig (11) shows
   The directed graph corresponding to this relation when a relation is symmetric an undirected graph can be used insted of the directed graph

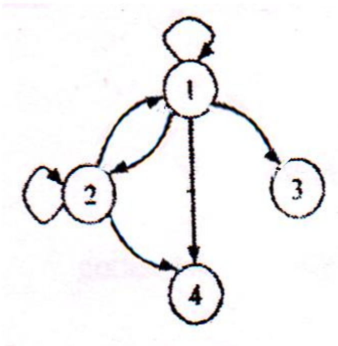


Fig (11) Directed graph)

2. Properties of relation on A Single Set

Now we shall see the fundamental properties of Relation defined on a set, that is R⊆ A x A we will review the properties such as reflexive Relation, symmetric relation, transitive relation, closure, equivalence, compatibility relation, pre-order relation and other relation in detail.

2.1. Fundamental properties

(a) Reflexive relation

If for all x ∈A, the relationxRx or $(x,x) \in R$ is established, we call it reflexive relation. The reflexive relation might be denoted as

x ∈A, → $(x,x) \in x$or $\mu_R(x,x)$= l; ∀ x ∈A

Where the symbol "→means "implication"

If it is not satisfied for some x ∈A, the relation is called "irreflexive" of it is not satisfied for all x ∈A the relation is " antireflexive".

When you convert a reflexive relation into the corresponding relation matrix, you will easily notice that every diagonal member is set to 1. A reflexive relation is often denoted by D.

5. Symmetric relation

For all x, y ∈A, if xRy = $y\ Rx, R$ is said to be a symmetric relation and expressed as

$$(x,y)\ \in R \rightarrow (y,x)\ \in R \quad or$$

$$\mu_R(x,x) = \mu_R(y,x), \forall x,y \in A$$

The relation is "asymmetric" or "nonsymmetric" when for some x, y ∈A, $(x,y) \in R$ and $(y,x)\ \notin R$ it is an antisymmetric relation if for all x, y ∈A, $(x,y) \in R$ and $(y,x)\ \notin R$

6. Transitive relation

This concept is achieved when a relation defined on A verifies the following property for all $x, y, z \in$ A

$$(x,y) \in R,\ (y,z) \in R \rightarrow (x,z)\ \in R$$

7. Closure

When relation R is defined in A, the requisites for closure are

1) Set A should satisfy a certain specific property.
2) Intersection between A's subset should satisfy the relation R.

The smallest relation R "Containing the specific property is called closure of R.

Example – 2.1

if R is defined on A, assuming R is not a reflexive relation then R = DUR contains R and reflexive relation. At this time, R is said to be the reflexive closure of R.

Example – 2.2 if R is defined on A, transitive closure of R is as follows (Fig 12) which is the same as R (reachability relation)

R^∞= RUR^2UR^3U……….

The Transitive closure R^∞ of R for A = { 1,2,3,4} and R = {(1,2), (2,3), (3,4), (2,1)} is

R^{∞}= { (1,1);(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,4)}

(Fig 12) explains this example

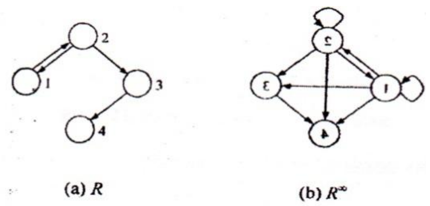


Fig 12 Transitive closure

2.2. Equivalence Relation

Definition (Equivalence relation) Relation (R) ⊆ A x A is an equivalence relation if the following conditions are satisfied

(i) Reflexive Relation

$$x \in A \rightarrow (x, x) \in R$$

(ii) Symmetric Relation

$$(x, y) \in R \rightarrow (y, x) \in R$$

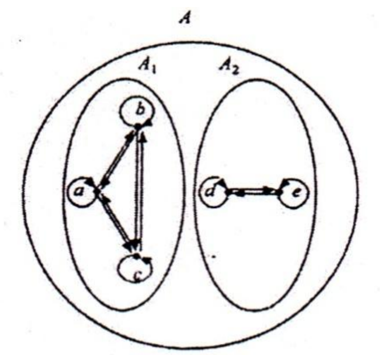
(iii) Transitive Relation

$$(x, y) \in R, (y, z) \in R \rightarrow (x, z) \in R$$

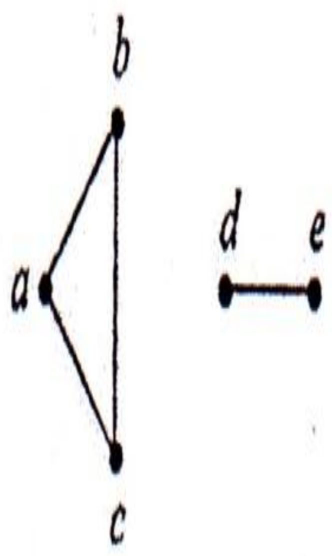
If an equivalence relation R is applied to a set we can perform a partition of A into n disjoing subsets A_1, A_2 ….. which are equivalence classes of R. At this time in each equivalence class, the above three conditions are verified.

Assuming equivalence relation R in A in given equivalence classes are obtained. The set of these classes is a partition of A

(a) Expression by set
(b) Expression by undirected graph



(a) Expression by set



(b) Expression by undirected graph

Fig 13 partition by equivalence relation by R and denoted as π(A/R) fig 13 shows the equivalence relation verified in A1 and A2

$$\pi(A/R) = \{ A1, A2 \} = \{ \{a,b,c\} , \{d,e\}\}$$

2.3. Compatibility Relation (Tolerance Relation)

Definition (Compatibility relation)

If a relation satisfies the following conditions for every $x, y \in A$ the relation is called compatibility relation.

1) Reflexive relation

   $x \in A \rightarrow (x, x) \in R$

2) Symmetric relation

   $(x, y) \in R \rightarrow (y, x) \in R$

If a compatibility relation R is applied to set A, we can decompose the set A into disjoint subsets which are compatibility classes in each compatibility relation on a set a gives a partition but the only difference form the equivalence relation is that transitive relation is not completed in the compatibility relation.

(a) Expression by set
(b) Expression by undirected graph

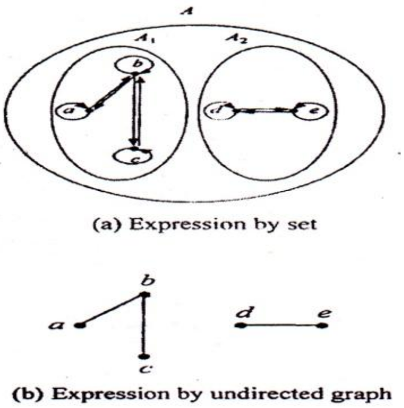


Fig 14 partition by compatibility relation

(Fig 14) describe a partition of set
A by a compatibility relation.
Here, Compatibiltity classes are
{a,b,c} and {d,e}

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