REPRESENTATION THEORY OF TOPOLOGICAL GROUPS

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Abstract:

In this paper, representation theory of topological groups these notes are intended to give an introduction to the representation theory of finite and topological groups. We have discuss to continuous homomorphism of a topological group into various groups of matrices. The device of mapping a topological group into a group of matrices transforms the study of abstract groups into that of "concrete ones". After establishing Schwr's lemma and the orthogonality relations, we shall show that, for a metrizable compact group, the set of all ortho-normal continuous complex valued functions are countable. We shall also establish the famous theorems of Peter-Weyl.

Keyword:

orthogonality relations, ortho-normal, Symmetric Group, invertiable, linear functional, bilinear Mapping, metrizable compact group.

1. Introduction:

We shall discuss continuous homomorphism of a topological group into various groups of matrices. The device of mapping a topological group into a group of matrices transforms the study of abstract groups into that of "concrete ones" (Husain, 1966, Lahiri, 1980). After establishing Schwr's lemma and the orthogonality relations, we shall show that, for a metrizable compact group, the set of all ortho-normal continuous complex valued functions are countable. We shall also establish the famous theorems of Peter-Weyl. We assume that the reader is only familar with the basics of group theory, linear algebra, topology and analysis. We begin with an introduction to the theory of groups acting on sets and the representation theory of finite groups, especially focusing on representations that are induced by actions. We then proceed to introduce the theory of topological groups, especially compact and amenable groups and show how the ”averaging” technique allows many of the results for finite groups to extend to these larger families of groups. We then finish with an introduction to the PeterWeyl theorems for compact groups.

Definition 1.1

We shall assume a topological group under our consideration as a locally compact, Hausdroff, second countable topological group. But we shall prove some results which will be true also for topological groups which are not necessarily second countable.

Any Hilbert space is assumed to be a separable complex Hilbert space. We use the symbol BL(H) to denote the set of bounded (≡ continuous) linear operators of H into itself. Also we use Aut (H) = GL(H) to denote the set of all bounded bijectve linear operators whose inverses are also bounded.

i.e., GL(H) = (A ∈ BL(H) : A^{-1} exists and A^{-1} ∈ BL(H)}.

Definition 1.2

A group homomorphism π : G → GL(H) such that g → π(g)v is continuous for all v ∈ H, is also called a representation of a topological group.

We then call x as a representation of G on the Hilbert space H.

Definition 1.3

The representation x is called a unitary operator for every g ∈ G. i.e.,

π(g) π(g)* = I = π(g)* π(g),

Where I is the identity operator or equivalently

(π(g)u, π(g)v) = (u, v), for all g ∈ G, u,v ∈ H,

Or || π(g)v || = || v ||.

Note 1.1

(i) If GL(H) is equipped with the operator norm topology then the continuity requirement in definition may demand the map $g \to \pi(g)$ from $G \to GL(H)$ be continuous i.e., to demand $g_n \to g$ implies

$|| \pi(g_n) - \pi(g)|| \to 0.$

Here, $|| A ||$ denotes the operator norm of the operator A.

(ii) Let H be a finite dimensional, then so is BL(H) and hence all the Hausdroff vector space topologies on BL(H).

(iii) We can define representation of G on any Banach space in the same manner

(iv) Let (π, H) be a representation of G on a Hilbert space H. Then for any compact subset $K \subseteq C$ and any fixed $v \in H$, we have $\sup_{g\in K} || \pi (g)v || \leq M_v$, where M_v depends on v. Therefore, by uniform boundedness principle $|| \pi (g) || \leq c$, c being a constant > 0 depending on K and $|| \pi (g) ||$ being the operator norm of π (g). In particular if G is a compact group, then there always exists a constant c > 0 such that $|| \pi (g) || \leq c$ for every $g \in G$.

Definition 1.4

Any representation (π, H) is called uniformly bounded if there exists a constant c > 0 such that $|| \pi (g) || \leq c$ for all $g \in G$.

Thus, a unitary representation of any group is uniformly bounded. Similarly, any representation of a compact group is uniformly bounded.

Example 1.1

(a) Consider the symmetric (=permutation) group s_n on n symbols {1, ... , n}. We may consider s_n as a topological group with discrete topology. We define $\pi: S_n \to GL (₵^n)$ by setting $\pi (\sigma)x = (x_{\sigma(1)} ...... x_{\sigma(n)})$ for $x = (x_i .... x_n) \in ₵^n$. Then π is a unitary representation of S_n on $₵^n$.

(b) We consider the group of all invertible n ⊗ n real (or complex) matrices G = GL(n, R) or GL (n, ₵) and define π : g → g where H = $₵^n$. Then π is a representation of G. Note that it is not unitary.

(c) let G be the set of unitary matrices of order n. i.e., G = U(n) = {A ∈ GL(n) = GL ($₵^n$) : AA* = 1}. Then G is a compact group. Also we take H = $₵^n$, and define π: G → GL(H) by π(g)v = g v, g ∈ G, v ∈ $₵^n$ where g x v denotes the usual matrix multiplication of the n x n matrix g and the column factor v. Then π is a unitary representation of U(n).

(d) Let G = ₵, the additive group of complex numbers H = $₵^2$. We define, $\pi(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ for every z ∈ ₵. Clearly, π(z) ∈ GL($₵^2$) = GL(2, ₵). Then π is a non-unitary representation of G.

(e) Let G = S^1 = {z ∈ ₵: | z | = 1}. Then S^1 becomes a group under multiplication and it is a compact group, with respect to the topology induced from ₵. We have for g ∈ G, g = $e^{i\theta}$, 0 ≤ θ ≤ 2π. Let H = ₵. Then GL (₵) = C*, the multiplicative group of non-zero complex numbers. Now we set for any n ∈z, χ_n ($e^{i\theta}$) = $e^{in\theta}$ ∈ ₵* = GL(₵). Then χ_n is a unitary representation of G on C.

(f) We consider G = U(n), the set of unitary matrices of order n. Let H = M (n, ₵) ≈ $₵^{n2}$. Then H has a natural inner product given by <x, y> = $\sum_{i,j} x_{ij}\, \bar{y}_{ij} = t_r\,(XY^*)$. We define π(g)x = g x g^{-1} for g ∈ G, x ∈ H. Then (π, H) is a unitary representation of G.

(g) Let g be a locally compact group and dg right Haar measure on G. Let H = L^2(G, dg) = L^2(G). Let C_o (G) be the subspace of continuous functions with compact support. Then C_o (G) is dense in L^2(G) We define (R (a)f)(g) = f(ga) for any f ∈ C_o (G) and a, g ∈ G. Then || R (a)f || = || R (a)f || = || f || for all a ∈ G. the norms here are all L^2 – norms.

$$|| R(a)f ||^2 = \int_G | (R(a)f)(g) |^2 \, dg$$

$$= \int | f(ga) |^2 \, dg$$

$$= \int | \bar{f}(g) |^2 \, dg \text{ since dg is right invariant.}$$

$= \| f \|^2$

Thus R(a) is a unitary and hence continuous operator on a dense subspace and hence R(a) extends as a unitary operator on $L^2(G)$.

Now, it is easy to check that R: G →BL ($L^2(G)$) is a homomorphism. To show that R is a (unitary) representation, we should check the continuity of the map g →R(g) f for any $f \in L^2(G)$. Let $\in > 0$. We choose $\Phi \in C_o(G)$ such that $\| f - \Phi \| < \in$. Since Φ is a continuous function with compact support, Φ is uniformly continuous.

Now given $\in > 0$, we choose a compact nhood of e in G such that $| \Phi(x) - \Phi(y)| < \in$ whenever $x \in yU$. Then we have, for x, y such that $x \in yU$

$\| R(x)f - R(y)f \| \leq \| R(x)f - R(y)\Phi \| + \| R(x)\Phi - R(y)\Phi \| + \| R(y)f - R(y)\Phi \|$

Now, $R(x)f - R(x)\Phi = R(x)(f-\Phi)$ and since R(x) is earlier shown to be unitary and $\| f - \Phi \| < \in$, we have the first and the last terms of the right side of the inequality above are $< \in$.

Also, the middle term:

$\| R(x)\Phi - R(y)\Phi \|^2 = \int | \Phi(gx) - \Phi(gy)|^2 \, dg$

But $gx \in gy\,U$. Thus

$\| R(x)\Phi - R(y)\Phi \|^2 \leq \in^2. M (\text{supp } \Phi)$.

Thus $\| R(x)f - R(y)f \|^2 \leq \eta$, for any pre assigned η.

This establishes the continuity of the representation R. R is called the right regular representation of G.

(h) Consider a finite group G with discrete topology on it. Then G becomes a compact group. Let H be the group algebra of G over ₵. i.e., h = ₵ [G]. Then H will consist element of the form $\alpha = \sum_{g\in G} \alpha_g g$, $\alpha_g \in$ ₵. Thus H is a finite dimensional vector space over ₵ with dim H = | G |, the number of elements in G. Now we define an inner product on H by setting

$(\alpha, \beta) = \frac{1}{|G|} \sum_g \alpha_g \bar{\beta}_g$, α, β ∈H.

Now set π (a) α = $\sum_g \alpha_g$ g a^{-1} where g a^{-1} is the multiplication of the two elements g and a^{-1} in G. Clearly, H consists of all (continuous) functions α : G → ₵, the identification being α(g) = α_g and α = $(\alpha_g)_{g \in G}$ is the | G | - tupelo of complex numbers.

Definition 1.5

Let W be a dimensional Hilbert space. Then G = GL (W) has a natural representation W given by g → g. Suppose we have two different sets of basis for W, say {e_i, ,….. e_n} and {f_i, ,….. f_n}. Then with respect to each of the basis we can write the linear operator g : W → W as matrices (g_{ij}) and (g'_{ij}). Thus the maps g → (g_{ij}) and g → (g'_{ij}) are two representations of G on $₵^n$. ($₵^n$ being identified with H via the basis {e_i} and {f_i} respectively.

Clearly the two representations are essentially the same. Thus we have following conditions:

Let (π, H) and (σ, V) be two representations of a group G. A bounded linear map A : H → V is called a G – map if

A(π (g)u) = σ (g)(Au)

for all u ∈ H and g ∈ V. we also say A inter twins the actions of G on H and V. The set of such G-maps id denoted by Hom_G (H, V).

We say (π, H) and (σ, V) are equivalent if there exists a G – map A such that

(i) A is one to one onto and

(ii) A^{-1} is continuous.

Definition 1.6

Let (π, H) be a representation of G and V be a (linear) subspace of H such that

π (g) V ⊆ V for all g ∈ G.

Then V is said to be π – stable or G-stable.

Note that the closure $\bar{V}$ of V in H is also π – stable.

Definition 1.7

An invariant subspace for π or G is a closed (linear) subspace V of G such that

$\pi(g) V \subseteq V$ for all $g \in G$.

Clearly, V = {0} and V = H are invariant subspaces. If H does not have any other invariant subspace, then (π, H) is a representation of G.

Note 1.2

If H is finite dimensional, any subspace is complete and hence closed. Hence any π – stable subspace is also π – invariant. If (π, H) is a representation of G, with $\dim H = n < \infty$, then we say π finite dimensional.

Example 1.2

(a) Any one dimensional representation of a group G is irreducible. Hence the representation of example 1.1 (e) is irreducible.

(b) The representation in (b), (c) and (e) of example 2.1 is irreducible.

Lemma 1.1

If (π, H) is a unitary representation of G and if $V \subseteq H$ is G – stable subspace, then $V^{\perp} = \{u \in H : (u, v) = 0 \text{ for every } v \in V\}$ is an invariant subspace.

Proof:

Clearly, $V^{\perp}$ is always a closed subspace. So it is sufficient to prove that whenever $u \in V^{\perp}$, $\pi(g)\, u \in V^{\perp}$ for any $g \in G$. Since π(g) is unitary we have $\pi(g)\, \pi(g)^* = \pi(g)^*\, \pi(g) = 1$ and hence $\pi(g^{-1}) = \pi(g)^*$. Since π is a group homomorphism

$\pi(e) = \pi(gg^{-1}) = \pi(g)\,\pi(g^{-1})$

Therefore $\pi(g)^{-1} = \pi(g^{-1})$.

Now if $v \in V$, $(\pi(g)u, v) = (u, \pi(g)^* v)$

$= (u, \pi(g)^{-1} v)$

$= (u, \pi(g^{-1}) v)$

Since $v \in V$ and V is invariant, $\pi(g^{-1}) v \in V$.

But the fact that $u \in V^{\perp}$ implies that $(\pi(g)u, v) = (u, \pi(g^{-1}) v) = 0$.

Thus $\pi(g)u \in V^{\perp}$, since $v \in V$ is arbitrary.

Definition 1.8

Two unitary representations (π, H) and (σ, V) of a group G are unitarily equivalent if there exists a G – intertwining operator $T: H \to V$ such that T is a liner isomorphism of H onto V and $(Tu, Tv) = (u, v)$ for all $u, v \in H$.

In this case T is called unitary equivalences.

Proposition 1.1

Two unitary representations (π, H) and (σ, V) are equivalent if, and only if, they are unitarily equivalent.

Proof:

Firstly, supposes $T: H \to V$ a G – map such that T^{-1} is also continuous. Then H and v will be endowed with two different inner products say < , > and (,) respectively corresponding to H and V.

Now let $P = T^*T : H \to H$. Then P is a positive operator on H with respect to < , > : $\langle T^*Tu, u \rangle = (Tu, Tu) > 0$ and is 0 if, and only if, $u = 0$ as T is one to one.

Also by a deduction of spectra theorem we know that P has a square root S in the sense that $S^2 = P$, S is positive, one to one and onto and such that $A : H \to H$ a bounded linear operator commutes with P if A commutes with S. Now let $B = TS^{-1}$. Then B is one to one onto and B, B^{-1} both is continuous. For $u \in H$, (Bu, Bu)

$= (TS^{-1}u, S^{-1}u)$

$= < T^* TS^{-1}u, S^{-1}u >$

$= < S^2 S^{-1}u, S^{-1}u >$

$= < Su, S^{-1}u >$

$= < u, S^* S^{-1}u >$

$= < u , u >$

Since $S^* = S$ so that $S^* S^{-1} = S S^{-1} = I$. Thus B is a unitary operator: $(H, < , >) \to (H, (,))$.

Also we have

$B \pi (g) = TS^{-1} \pi (g) = T\pi (g) S^{-1}$

$= \sigma (g) TS^{-1} = \sigma (g) B.$

Thus, B is a unitary equivalence, hence the proof.

Now we shall discuss construction of new representation from the given ones.

Definition 1.9

Direct sum

Let (π_i, H_i), i = 1, 2 be representations of a group G. the inner products on H_1 are denoted by the same symbol < , >. Let $H = H_1 \oplus H_2$ be the direct sum of Hilbert spaces. The inner product on H is given by $< (u_1, v_1), (u_2, v_2) > = < u_1, u_2> + <v_1, v_2>$ where $u_1 \in H_1$, $v_1 \in H_2$.

Let $\pi(g) = \pi_1(g) \oplus \pi_2(g)$ so that $\pi(g)(u, v) =$

$(\pi_1(g)u, \pi_2(g)v)$, $u \in H_1$, $v \in H_2$. Then (π, H) is representation of G, called the direct sum of (π_1, H_1) and (π_2, H_2). It is unitary if (π_1, H_1) is unitary for every i.

Definition 1.10

Contra-gradient

Let (π, H) be a representation of G. Let H^* be the dual of H. i.e., the space of continuous linear functionals on H. We define

$(\pi^*(g)u)(v) = u(\pi(g^{-1})v)$ for $u \in H^*$, $v \in H$, $g \in G$.

Then (π^*, H^*) is a representation of G and is called contra gradient of (π, H).

Definition 1.11

Tensor Products

Let Bil (V, W) denote the set of all bilinear maps $B : V \times W \rightarrow$ i. e., for every fixed v, (respe. w) the map $w \rightarrow B(v, w)$ (Resp. $V \rightarrow B(V, U)$ is linear. Then $V \otimes W$ is by definition Bil (V^*, W^*). For $v \in V$, $w \in W$, we let $v \otimes w$ denote an element of $V \otimes W$ defined by the bilinear map

$\beta_{v \otimes w}(v^*w^*) = v^*(v)\, w^*(w)$,

$v^* \in v^*$, $w^* \in w^*$ and $\beta_{v \otimes w} \in$ Bil $(V^* \times W^*)$

If $\{v_i\}$ is a basis for $V \otimes W$. Thus $V \otimes W$ is a vector space over C of dimensdion = dim V x dim W.

There is a most natural isomorphism between Bil (V^*, W^*) and Hom (V^*, W^*), the space of linear maps from V^* to W given by as follows:

Let B ∈ Bil (V* x W*) we want an A ∈ Hom (V* x U). Thus given v* ∈ V*, Av* must be an element of W. Since $(W^*)^A$ = w, Av* is uniquely determined if we know what is its action on w* ∈ W*. So we want to define (Av*) (w*) ∈ C. What is more natural than setting (Av*) (w*) = B (v*, w*)? Obviously we can reverse the process i.e., A ∈ Hom (V*, W) set B (v*, w*) = (Av*) (w*). It can be easily checked that the above definitions are meaningful in the sense that given B ∈ Bil and A ∈ Hom (V*, W) and conversely and that the above correspondence B → A and A →B are linear.

Note 1.3

Since V ⊗ W = Bil (V*, W*), we have V ⊗ W is isomorphic to Hom (V*, W) in a natural way. Some self consequence of this isomorphism is

(a) V ⊗ C = Bil (V*, C*) = Hom (V*, C)

= (V*)* = V.

(b) V* ⊗ V = Bil (V*)*, V*)

= $Hom_¢$ (V*, V)

= $Hom_¢$ (V, V), where

We use the fact v** is isomorphic naturally to V. By definition $Hom_¢$ (V, V) is the space of linear endomorphism of V to itself.

Now we consider this isomorphism between V* ⊗ V and $Hom_¢$ (V, V) a little more closely. Firstly, if $\{e_i\}$ is a basis for V and $\{u_i\}$ a basis for V*, then the elements $\{u_i - \in_j : 1 \leq i, j \leq \dim V\}$ form a basis for V* ⊗ V. Thus any element of V* ⊗ V is a linear combination of the from $\Sigma a_{ij}\, u_i \otimes e_j$.

Thus, given $u_i \otimes e_j$ ∈ V* x V.

We want to define $A = Au_i \otimes e_j \in Hom_¢ (V, V)$.

Therefore, we must say what the element $Au_i \otimes e_j$ (v) is for any $v \in V$. The only obvious choice is $Au_i \otimes e_j (v) = u_i (v) \overline{e_j}$, as $u_i (v) \in Hom_¢ (V, V)$.

In general we choose $\{u_i\} = \{\in_i\}$, the basis of V* dual to (e_i) i.e., $\in_I (e_j) = \delta_{ij}$. (when δ_{ij} is the Kronecker delta : $\delta_{ij} = 0$ if $i \neq j = 1$ if $i = j$). Then the element $\Sigma a_{kj} \in_k \otimes \in_j$ corresponds to a linear map $A : V \to V$ given by

$$(\sum_{k,j} a_{kj} \in_k \otimes e_j)(e_i) = \sum_j \sum_k a_{ki} \in_k (e_i) e_j$$

$$= \sum_j (\sum_k a_{kj} \delta_{ki}) e_j = \sum_j a_{ij} e_j$$

Thus, the linear operator $A: V \to V$ has a matrix representation (a_{ij}) with respect to the basis $\{e_i\}$. In particular the identity $I: V \to V$ corresponds to $\Sigma_{ij} \in u_i \otimes e_j \in V^* \otimes V$ for any choice of basis $\{e_i\}$ of V and the corresponding dual basis $\{\in_i\}$ of V*.

Definition 1.12

If V and W are two finite dimensional inner product spaces, then there is a natural inner product on $V \otimes W$ given as follows:

Let $\{v_i\}$ and $\{w_j\}$ are sets of ortho-normal basis of V and W respectively then $\{v_i \otimes w_j\}$ be an ortho-normal basis of $V \otimes W$. Then the inner product of $V \otimes W$ is $<v_1 \otimes w_1 , v_2 * w_2> = <v_1 , v_2> < w_1, w_2>$ in an obvious notion.

Definition 1.13

Let (π_i, H_i) $i = 1, 2$ be any two finite dimensional representations of G. we define,

$\pi (g) = \pi_1(g) \otimes \pi_2(g)$

where $\pi_1(g) \otimes \pi_2(g) (v_1 \otimes v_2) = \pi_1(g)v_1 \otimes \pi_2(g)v_2$

for $v_i \in H_i$. Then,

$(\pi, H) = (\pi_1 \otimes \pi_2, H_1 \otimes H_2)$ is a representation of G and it is unitary if each π_1 is unitary.

Definition 1.14

Let $(\pi_i, H_i)_{i \in I}$ be a family of unitary representations of G. Let $H_o = \bigoplus_{i \in I} H_i$ be the vector space direct sum of H_i. Then any element v of H_o is $v = (v_i)_{i \in I}$ with $v_i = 0$ except for finitely many i. Now H_o has natural inner product given by for $u = (u_i)$, $v = (v_j)$. $<u, v> = \sum_{i \in I} <u_i, v_i> <u_i, v_i>$.

This definition <u, v> is finite summation. Also H_o is complete only when I is finite. Thus, it is only a pre-Hilbert space. We use H to denote the completion of H_o with respect to the above inner product and write $H = \overline{\otimes}\ H_i$; H can also be described as the space of elements $v = (V_i)$ such that the sum $\sum || v_i ||^2 < \infty$. Here

$|| v ||^2 = \sum || v_i ||^2$, for $v \in H$.

For example, $H_n = C$, for n = 1, 2, ... Then $H = \oplus H_n$ is the set of sequence $(V_n)^{\infty}{}_{n=1}$, $v_n \in C$ which are such that $v_n = o$ for n sufficiently large. Then

$H = \overline{\oplus}\ H_n = 1^2$.

Now it is clear that $\pi(g)v = (\pi_1(g)v_i)$

Where $v = (v_i) \in H$. It can be seen that π is a unitary representation of C.

Definition 1.15

We define the multiplicity of an irreducible representation (π_λ, H_λ) of G in a possibly infinite dimensional representation (π, H) of G in the following way:

$m_\lambda(\pi)$ = the multiplicity of (π_λ, H_λ) in (π, H)

$= \dim Hom_G (H_\lambda, H)$

$= \dim Hom_G (H, H_\lambda)$

Theorem 1.2

(Peter Weyl theorem I)

Let $\widehat{G}$ denote the set of classes of equivalent representations of a compact group G. Let, for $\lambda \in \widehat{G}$, $H(\lambda)$ denote the subspace (of dimension $d(\lambda)^2$) spanned by the matrix entries of the representation $(\pi_\lambda, H_\lambda) \in \lambda$. Then each H_λ is an invariant subspace under the (right) regular representation R and $L^2(G) = \overline{\bigoplus}_{\lambda \in \hat{G}} H(\lambda)$. [Hilbert space direct sum]. Also, if we choose a basis $\{e_i\}$ of H_λ then the subspace $H(\lambda) = \sum_j ₵\, \Phi_{ij}^{\lambda}$ is invariant under R for each i such that $(R, H(\lambda)_i)$ is equivalent to (π_λ, H_λ) occurs in $(R, L^2(G))$ with multiplicity $d(\lambda)$. Also the set $\{\sqrt{d(y)}\, \Phi_{ij}^{\lambda}, 1 \le i,j \le d(\lambda), \lambda \in \hat{G}\}$ is an orthonormal basis for $L^2(G)$.

Proof:

For a $\in G$, $x \in G$, we have

$$R(a)\, \Phi_{ij}^{\lambda}(x) = \Phi_{ij}^{\lambda}(xa) = \sum_k \Phi_{ij}^{\lambda}(x)\Phi_{ij}^{\lambda}(a)$$

$$= \sum_k \Phi_{kj}^{\lambda}(x)\Phi_{ik}^{\lambda}(a)$$

Or, $R(a)\, \Phi_{ij}^{\lambda} = \sum_k \Phi_{kj}^{\lambda}(a)\Phi_{ik}^{\lambda}$ for 1 fixed

Or, $(R(a))\, \Phi_{kj}^{\lambda} = \pi_\lambda(a)_{kj}$ with respect to the basis Φ_{ik}^{λ} of $H(\lambda)_i$.

Thus, $(R, H(\lambda)_i)$ is equivalent to (π_λ, H_λ).

The set $\{\sqrt{d(y)}\, \Phi_{ij}^{\lambda}\}$ is an orthonormal set by Schur's orthogonality relations. Now the proof of all the statement will be completed if we show that set $\{\sqrt{d(y)}\, \Phi_{ij}^{\lambda}, 1 \le i,j \le d(\lambda), \lambda \in \hat{G}\}$ is a complete basis.

Let H_0 be the closure of the subspace appaned by the above set. Then H_0 is an invariant subset for the regular representation R. If $H_0 = L^2(G)$ then the result follows. If $H_0 \neq L^2(G)$, then $H_0^{\perp} \neq \circ(0)\circ$. Since R is unitary $H_0^{\perp}$ is invariant under R, So by theorem $(R, H_0^{\perp})$ is a Hilbert space direct sum of irreducible (finite dimensional) representations of G. In particular there is non-zero finite dimensional subspace $V \subseteq H_0$ such that (R, V) is irreducible. Let $0 \neq v \in V$ we take

$$u(x) = (R(x)v, v)_{L^2(G)} = (R(x)v, v)$$

$$= \int_G R(x)v(y)\bar{v}(y)\, dy$$

$= \int v(yx)\, \bar{v}(y)\, dy,$

Where u is a matrix entry, is a continuous function on G. Since

$u(e) = (R(e)v, v) = ||v||^2 \neq 0.$

u is a non zero function. We claim that $u \in H_0^{\perp}$, which will complete the proof the theorem.

Since u is a matrix entry $u \in H_0$ and this is a contradiction as $u \neq 0$. We now rove our claim. Let Φ_{ij}^{λ} be a matrix entry, then

$$\int u(x)\, \overline{\Phi_{ij}^{\lambda}(x)}\, dx$$

$$= \int (\int v(yx)\bar{v}(y)dy)\, \overline{\Phi_{ij}^{\lambda}(x)}\, dx$$

$$= \int \int v(z)\bar{v}(y)\, \overline{\Phi_{ij}^{\lambda}(y^{-1}z)}\, dydz,$$

by Fubini and then by a change of variable z = yx.

$$= \sum_k \int \int v(z)\bar{v}(y)\, \overline{\Phi_{ik}^{\lambda}(y^{-1})\Phi_{kj}^{\lambda}}\, (z)\, dydz$$

$$= \sum_k \int v(z)\, \overline{\Phi_{kj}^{\lambda}(z)}\, dz\, \bar{v}(y)\Phi_{kj}^{\lambda}(y)dy$$

$= 0$, since $v \in H_0^{\perp}$

Hence the claim and the theorem are proved.

Note 1.4

(i) An analogous version of Peter Weyl theorem holds if we consider (L, L^2 (G)) and set

$V_j^{\lambda} = \sum_i ₵\, \Phi_{ij}^{\lambda}$ and see that

$L(a)\, \Phi_{ij}^{\lambda} = \sum_k ₵\, \Phi_{ij}^{\lambda}(a^{-1})\, \Phi_{kj}^{\lambda}$ so that $(L, V_j^{\lambda}) \in \lambda^*$, the contra-gradient of λ . Also as λ varies over $\widehat{G}$, λ^* varies over $\widehat{G}$.

(ii) We claim that Peter Weyl theorem implies $(\pi, L^2(G)) = \overline{\bigoplus}_{\lambda \in \widehat{G}} (\pi_\lambda^* \otimes \pi_\lambda,\ H_\lambda^* \otimes H_\lambda)$ are representation spaces for G x G. To prove the claim we set a G x G map from $H_\lambda^* \otimes H_\lambda$ to L^2 (G).

Also for $u \otimes v \in H_\lambda^* \otimes H_\lambda$, we define

$\Phi_{u,v}(g) = u(\pi_\lambda(g)v)$. Now it can be easily seen that

$$\Phi_{\pi\lambda^*(a)u\otimes\pi\lambda(b)v}(g) = \Phi_{u,v}(a^{-1} g\, b)$$

$$= (\pi(a, b)\, \Phi_{u,v}(g))$$

i.e., the map $H_\lambda^* \otimes H_\lambda \rightarrow H(\lambda)$ given by $u \otimes v \rightarrow \Phi_{u,v}$ is a G x G map.

(iii) Since $H_\lambda^* \otimes H_\lambda \backsimeq End(H_\lambda) = Hom_{\mathbb{C}}(H_\lambda, H_\lambda)$. We make End (H_λ) a G x G representation space via this identification i.e., by setting

$\rho_\lambda(x,y)A = \pi_\lambda(x\text{-}1)A\,\pi_\lambda(y)$ for

$x, y \in G$, $A \in End(H_\lambda)$.

Definition 1.16

We denote the diagonal subgroup $\{(g, g): g \in G\}$ by $\Delta(G)$.

Clearly, $\Delta(G)$ is topologically isomorphic to G. We also have $G \times \frac{G}{\Delta(G)} \simeq G$ as G x G homogenous spaces.

We say that an irreducible representation (ρ, V) of G x G has a $\Delta(G)$ - fixed vector if there is a non-zero $v \in V$ such that $\rho(g, g)\, v = v$ for all $g \in G$. it can be easily seen that $(\pi_\sigma \otimes \pi_\lambda,\ H_\sigma \otimes H_\lambda)$ has a $\Delta(G)$ –fixed vector if $(\pi_\sigma, H_\sigma) = (\pi^*_\lambda, H^*_\lambda)$ i.e., $(\pi_\sigma, H_\sigma) \in \lambda^*$. Thus we may reinterpret the above decomposition of $L^2(G \times \frac{G}{\Delta(G)})$ into the Hilbert space direct sum of all irreducible representation of G x G that have a $\Delta(G)$ –fixed vector.

Now we shall discuss Fourier transform on compact group.

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